

Introduction to category theory

Sylvain Douteau

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Chapter 1

Categories

Introduction

These notes are written specifically to accompany the class "introduction to category theory" from the master LMFI of Université Paris Cité. The notes themselves are mostly self contained, and should be accessible to anyone who knows set theory, and with some basic notions of algebra for some examples (some knowledge about groups and vector spaces is enough).

Many exercises are spread along the text. While it may be too time consuming to do all of them, the reader is highly encouraged to do at least a few of them in each chapter. They should all be doable with only the knowledge contained in the notes up to the point that they appear. Exercises appearing in proof tend to be more time consuming than others, but mainly due to the repetitiveness of checking definitions.

Any reader who would want to compare to a classical reference should first refer to [Mac71]. An important difference in point of view is that, in these notes, we do not consider (nor even define) non-locally small categories. Those notes are constructed as follows.

Chapter 1 is concerned with categories themselves. In this chapter, we define categories, and mention a few elementary constructions that can be done with just categories.

Chapter 2 introduces functors and natural transformations. After having defined categories, we will see how we can relate them to each other. In particular, we will introduce the category of all small categories (Definition 2.1.3). We conclude the chapter with the notion of equivalence of categories, allowing us to identify categories (Section 2.3).

In Chapter 3, we introduce adjunction, and study some important examples. (See Example 3.2.4, detailing the adjunction between the category of small categories and the category of directed graphs). Adjunctions will appear throughout the rest of those notes, and we take some time to prove that adjoint functors are essentially unique (Section 3.3) and to prove a particular interaction between

adjunctions and commutative diagrams (Section 3.4).

In Chapter 4, we finally introduce limits and colimits of diagrams. We first take some time to define and study limits in details (Sections 4.1, 4.2 and 4.3). Then, in section 4.4, we explain how to define colimits by dualizing the definition of limits, and recall the results that are obtained for free this way. We conclude with Section 4.5 which details a few concrete computations.

1.1 What is a category?

The central notion in this course is that of a category. Before giving a definition, let us give some motivation coming from the idea of **universal properties**.

Motivation 1.1.1 (Product of sets). Let X and Y be two sets. If we want to define **the** product $X \times Y$, we need to give a specific **model** for it. In classical set-theory, we may choose to define it as follows

$$X \times Y = \{(a, b) \mid a \in X, b \in Y\}$$

where, (a, b) is represented by the set $\{\{a\}, \{a, b\}\}$. Now, observe that this new set comes with two maps

$$X \xleftarrow{\text{pr}_X} X \times Y \xrightarrow{\text{pr}_Y} Y \quad (1.1)$$

Indeed, given a pair (a, b) , one can recover the element $a = \text{pr}_X(a, b) \in X$ and the element $b = \text{pr}_Y(a, b) \in Y$. Furthermore, the data of $\text{pr}_X(c)$ and $\text{pr}_Y(c)$ completely determines $c \in X \times Y$. In fact, we can say that the set $X \times Y$, together with those two maps, satisfy a certain **universal property**. Let us make this precise. Assume that we are given the data of two maps $f: Z \rightarrow X$ and $g: Z \rightarrow Y$:

$$\begin{array}{ccc} & Z & \\ f \swarrow & & \searrow g \\ X & & Y \end{array} \quad (1.2)$$

Then, we can define, for any $z \in Z$, $h(z) = (f(z), g(z)) \in X \times Y$. This gives a map $h: Z \rightarrow X \times Y$. But now, observe that for any $z \in Z$, we have the relations $\text{pr}_X(h(z)) = f(z)$ and $\text{pr}_Y(h(z)) = g(z)$. This can be summarized by saying that the following diagram is **commutative**, meaning that any two ways of going from a set to another set by following arrows in the graph must give the same result.

$$\begin{array}{ccccc} & & Z & & \\ & f \swarrow & \downarrow h & \searrow g & \\ X & & X \times Y & & Y \\ & \xleftarrow{\text{pr}_X} & & \xrightarrow{\text{pr}_Y} & \end{array} \quad (1.3)$$

And finally, observe that $h: Z \rightarrow X \times Y$ was the **unique** map with the property of making the diagram (1.3) commutative. Thus, we can make explicit the **universal property** that $X \times Y$ satisfy. It is defined, together with two maps,

giving a diagram of the form (1.1), and for any diagram of the same **shape**, (1.2), there exist a unique map, h , making the diagram (1.3) commutative.

In fact, this universal property is enough to **define** the product $X \times Y$ in the following sense. Consider some other model of the product of sets. For example, set

$$X \times' Y = \{(a, b)' \mid a \in X, b \in Y\},$$

where $(a, b)' = \{\{a\}, \{b, \emptyset\}\}$. Clearly, $X \times Y$ and $X \times' Y$ are not **equal**, since they do not have the same elements. However, note that $X \times' Y$ can also be equipped with two projections

$$X \xleftarrow{\text{pr}'_X} X \times' Y \xrightarrow{\text{pr}'_Y} Y$$

Furthermore, the same reasoning as before shows that for any diagram of the form (1.2), there exists a unique map $h': Z \rightarrow X \times' Y$ making the following diagram commutative

$$\begin{array}{ccccc} & & Z & & \\ & f \swarrow & \downarrow h & \searrow g & \\ X & \xleftarrow{\text{pr}'_X} & X \times' Y & \xrightarrow{\text{pr}'_Y} & Y \end{array}$$

In particular, $X \times' Y$ satisfies the **same universal property** as $X \times Y$

Exercise 1. Use the universal property of the product to prove that $X \times Y$ and $X \times' Y$ are in bijection. Convince yourself that among all possible bijections, one of them (the obvious one) is the *correct* bijection.

Motivation 1.1.2. What we have seen is that, up to some bijection, the product of X and Y is determined by the **universal property** it satisfies. We will see later, in chapter 4, that there is indeed a **unique** bijection between two possible models that is better than the other. In a certain sense, we do not even **need** a specific model for the product, since we know everything there is to know about it (or rather about maps going into it) through its universal property. This is the core philosophy of category theory. We will no longer care about the *nature* of mathematical objects, but only about how they relate to one another. In that respect, we will forget about equality between objects, and instead consider two objects to be the same, interchangeable, **isomorphic**, if they *relate to other objects in the same way*. This can happen, as we have seen, if they satisfy the same universal property. This point of view puts the entire focus on **morphisms** between objects, and leads to the notion of a category.

Definition 1.1.3. A category, $\mathcal{C}$ is the data of

- a collection of **objects** (a **class**), denoted $\text{obj}(\mathcal{C})$,
- for any two objects in $\mathcal{C}$, X and Y , a **set of morphisms** from X to Y , $\text{Hom}_{\mathcal{C}}(X, Y)$. Instead of writing $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, we will often simply say that $f: X \rightarrow Y$, is a morphism in $\mathcal{C}$. Morphisms are also often called **arrows** or **maps**.

- For any object of $\mathcal{C}$, X , a chosen morphism, $\text{Id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$ called the **identity** of X ,
- for any triple of objects in $\mathcal{C}$, X, Y, Z , a **composition law**, i.e. a function

$$\begin{aligned} - \circ - : \text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) &\mapsto \text{Hom}_{\mathcal{C}}(X, Z) \\ (g, f) &\mapsto g \circ f \end{aligned}$$

diagrammatically, this means that for any pair of **composable morphisms**, f and g , represented in the following diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & \searrow \text{---} & & \nearrow \text{---} & \\ & & g \circ f & & \end{array}$$

We have **chosen** some specific morphism $g \circ f : X \rightarrow Z$, represented as a dashed arrow in the above diagram.

This data must satisfy two axioms :

- **associativity of composition**: for any triple of composable arrows $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $h : Z \rightarrow W$, we must have

$$h \circ (g \circ f) = (h \circ g) \circ f \in \text{Hom}_{\mathcal{C}}(X, W)$$

- **identity**, for any arrow $f : X \rightarrow Y$ in $\mathcal{C}$, we must have

$$f \circ \text{Id}_X = f \quad \text{Id}_Y \circ f = f$$

Example 1.1.4. Let us now list a few illuminating examples of categories.

1. Set, the category of sets.

- Its objects are the sets. (Note that this collection of objects does not form a set itself, but instead, a *class*).
- Given two sets X and Y , $\text{Hom}_{\text{Set}}(X, Y)$ is the set of set-theoretic functions from X to Y
- Given a set X , the identity of X , $\text{Id}_X \in \text{Hom}_{\text{Set}}(X, X)$ is the identity function, sending any element of X to itself, i.e. $\text{Id}_X(x) = x$ for all $x \in X$.
- The composition of two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ is defined as the function $g \circ f$ sending $x \in X$ to $g(f(x)) \in Z$.

What we know of set-theory tells us that this composition is associative and that the identity is a neutral element for the composition.

2. We may consider **sub-categories** of $\underline{\text{Set}}$ (see section 1.4). For example, $\underline{\text{FinSet}}$, the category of finite sets. Its objects are the finite sets, and given X and Y two finite sets, we may define $\text{Hom}_{\underline{\text{FinSet}}}(X, Y) = \text{Hom}_{\underline{\text{Set}}}(X, Y)$. Then, we can define its compositions, and the identities as those obtained by **restricting** those of the category $\underline{\text{Set}}$.
3. Another interesting sub-category of $\underline{\text{Set}}$ would be $\underline{\text{Set}}^{\text{bij}}$. Its objects are the same as those of $\underline{\text{Set}}$, but given two sets X and Y the morphisms from X to Y are given by $\text{Hom}_{\underline{\text{Set}}^{\text{bij}}}(X, Y) = \{f \in \text{Hom}_{\underline{\text{Set}}}(X, Y) \mid f \text{ is a bijection}\}$. Since Id_X is a bijection, given any set X , we still have identities in $\underline{\text{Set}}^{\text{bij}}$. In addition, since the composition of two bijections is still a bijection, we can also restrict the composition in $\underline{\text{Set}}$ to $\underline{\text{Set}}^{\text{bij}}$.
4. $\underline{\text{Grp}}$, the category of groups.
 - its objects are the groups,
 - given G and H two groups, the set of morphisms $\text{Hom}_{\underline{\text{Grp}}}(G, H)$ is the set of group homomorphisms from G to H .
 - the identity of a group G is the identity homomorphism,
 - composition is that of group homomorphisms.

Once again, we can deduce from what we know of sets and functions between sets that the axioms of a category are satisfied.

5. Given a field k (for example, $k = \mathbb{R}$), we can consider the category $\underline{k\text{-Vect}}$ of k -vector spaces, whose objects are k -vector spaces, and whose morphisms are k -linear maps. Once again, identity and compositions are clear from context, and it is easy to check that the axioms are satisfied.
6. Contrast the above (slightly repetitive) definitions with the following. Consider $\underline{k\text{-Mat}}$ the category, such that
 - its object are the integers $n \in \mathbb{N}$,
 - given $m, n \in \mathbb{N}$, we define

$$\text{Hom}_{\underline{k\text{-Mat}}}(m, n) = M_{m,n}(k),$$

the set of $m \times n$ matrices with coefficients in the field k .

- Given, $n \in \mathbb{N}$, the identity $\text{Id}_n \in \text{Hom}_{\underline{k\text{-Mat}}}(n, n)$ is the identity matrix of size n , (with 1's on the diagonal and 0's everywhere else).
- the composition of $N \in \text{Hom}_{\underline{k\text{-Mat}}}(m, n)$ with $N' \in \text{Hom}(n, p)$ is given by the product of matrices $N'N \in \text{Hom}_{\underline{k\text{-Mat}}}(m, p)$

Exercise 2. Check that $\underline{k\text{-Mat}}$ satisfies the two axioms of a category.

7. Consider now some pre-ordered set P . i.e. some set P and some relation on P , " $\leq$ ", assumed to be transitive and reflexive. Now, we can define a category $\underline{P}$ where

- its objects are the elements of P .
- given $p, q \in P$, $\text{Hom}_{\underline{P}}(p, q)$ is either a singleton, if $p \leq q$ or the empty set if $p \not\leq q$.
- Given $p \in P$, Id_p is the unique element in $\text{Hom}_{\underline{P}}(p, p)$.
- given $p, q, r \in P$ and two arrows $f: p \rightarrow q$ and $g: q \rightarrow r$, $g \circ f: p \rightarrow r$ is the unique element in $\text{Hom}_{\underline{P}}(p, r)$. Indeed, this makes sense since the existence of f and g implies that $p \leq q$ and $q \leq r$, hence, transitivity of the order relation means that $p \leq r$. In turn, there must indeed be a unique element in $\text{Hom}_{\underline{P}}(p, r)$.

Now, to see that the axioms are satisfied, observe that when compositions exist, they always return an element of a singleton. And any two elements produced in a singleton must be equal, which gives the needed equalities.

8. In the same vein as above, we can associate a category $\underline{M}$ to any monoid M .

- it has a single object $*$,
- $\text{Hom}_{\underline{M}}(*, *) = M$
- $\text{Id}_* = e$, the neutral element of M ,
- given $a: * \rightarrow *$ and $b: * \rightarrow *$ define the composition $b \circ a$ as the product $ba \in M$.

The axioms of a category then follow from those of a monoid.

9. But we may also define a category "by hand". For example, the empty category $\underline{\emptyset}$, with no objects and no morphisms.
10. The category $\underline{0}$ with a single object $*$, and a single morphism $\text{Id}_* \in \text{Hom}_{\underline{0}}(*, *)$.
11. the category $\underline{1}$ with two objects, A and B and exactly three morphisms $\text{Id}_A: A \rightarrow A$, $\text{Id}_B: B \rightarrow B$ and $f: A \rightarrow B$. One can check that there is only one possible way to define the composition.

Remark 1.1.5. The category we have seen above can be roughly separated into three kinds. First, there are categories containing information about all objects of a certain kind we may be interested in. We will often call those category **of objects**. Those could be the category of sets, the category of groups, or any other kind of mathematical structure. Those are examples 1, 2, 3, 4 and 5. Note that in the case of example 3, we may instead think of it as a category of a certain type of morphism, namely, the category of bijections (or to be more specific, of set-bijections). Then, there are examples of mathematical objects than can be **encoded as categories themselves**, such as pre-ordered sets or monoids, in examples 7 and 8. We may refer to one of those as "the pre-order P , *seen as a category*". Finally, we may consider elementary categories, with few objects and morphisms, where everything is defined by hand, such as the case of 9, 10

and 11. This "classification" may be helpful in navigating examples initially, though keep in mind that it is not exhaustive. Indeed, example 6 is not of any of the three kinds. As we keep seeing more involved examples, one may want to think of a fourth kind, of *abstract* categories. On the other hand, observe that examples 9, 10 and 11 are just particular cases of 7, which illustrates that while those considerations may be helpful in parsing examples, they do not exactly reflect a concrete classification.

Exercise 3. Consider the category $\underline{\text{Rel}}$ whose objects are sets, and whose morphisms are relations. More explicitly, for X, Y two sets, we define

$$\text{Hom}_{\underline{\text{Rel}}}(X, Y) = \{R \subset X \times Y\}$$

The composition of $R: X \rightarrow Y$ with $S: Y \rightarrow Z$ is defined as

$$S \circ R = \{(x, z) \mid \exists y \in Y, (x, y) \in R \text{ and } (y, z) \in S\}$$

1. find the formula for $\text{Id}_X \in \text{Hom}_{\underline{\text{Rel}}}(X, X)$, and show that $\underline{\text{Rel}}$ is a category.
2. Does there exist a category with the same objects and morphisms as those of $\underline{\text{Rel}}$, but with a different composition law (and identities) ? (Hint: consider negations of relations).

1.2 Small categories and alternative definitions

We will often be interested in the slightly more restrictive notion of **small categories**.

Definition 1.2.1. A category $\mathcal{C}$ is **small** if its class of objects, $\text{obj}(\mathcal{C})$ is actually a set.

In Example 1.1.4, the examples 6 through 11 were small categories. Note that very often, categories containing all objects with a certain mathematical structure will **not** be small. However, when available, the set-theoretical point of view is very convenient. See as an illustration the following alternative definition. Before giving this alternative definition, let us recall the definition of a directed graph.

Definition 1.2.2. A directed graph, G is the data of two sets,

- $V(G)$, the set of vertices of G ,
- $E(G)$, the set of edges of G .

Together with two maps, called source and target, $s, t: E(G) \rightarrow V(G)$ sending an edge to its starting point and endpoint respectively.

Alternative Definition 1.2.3. A small category $\mathcal{C}$ is a directed graph, together with the data of

- a choice of distinguished loop for each vertex, $\text{Id}: V(\mathcal{C}) \rightarrow E(\mathcal{C})$, such that, for any $X \in V(\mathcal{C})$, $s(\text{Id}_X) = t(\text{Id}_X) = X$.
- a partially defined map

$$\begin{aligned}
 - \circ - &: E(\mathcal{C}) \times E(\mathcal{C}) \rightarrow E(\mathcal{C}) \\
 (e_1, e_2) &\mapsto \begin{cases} e_1 \circ e_2 & \text{if } t(e_2) = s(e_1) \\ \text{undefined} & \text{if } t(e_2) \neq s(e_1) \end{cases}
 \end{aligned}$$

such that the partially defined composition is associative, and such that Id_X is a neutral element for the composition, for all $X \in V(\mathcal{C})$.

To see that the above is indeed equivalent to Definition 1.2.2, define $\text{obj}(\mathcal{C}) = V(\mathcal{C})$, and for all $X, Y \in \text{obj}(\mathcal{C})$, define $\text{Hom}_{\mathcal{C}}(X, Y) = \{f \in E(\mathcal{C}) \mid s(f) = X, t(f) = Y\}$.

Exercise 4. Check that the definitions 1.2.2 and 1.2.3 are indeed equivalent.

Remark 1.2.4. In a small category, $\mathcal{C}$, we allow ourselves to write statements such as $X = Y$ or $X \neq Y$ for X, Y two objects of the category. Indeed, such checks are necessary to define the partial composition in Alternate Definition 1.2.3. This is meaningful because in that case, X and Y are elements of a common set, $\text{obj}(\mathcal{C})$.

Exercise 5. For each of the following graphs, find possible identities and composition laws making them into small categories or show that it cannot be done.

$$A \tag{1.4}$$

$$f \hookrightarrow A \tag{1.5}$$

$$f \hookrightarrow A \rightrightarrows g \tag{1.6}$$

$$f \hookrightarrow A \begin{matrix} \xrightarrow{g} \\ \xleftarrow{h} \end{matrix} B \rightrightarrows i \tag{1.7}$$

$$f \hookrightarrow A \begin{matrix} \xrightarrow{g} \\ \xleftarrow{h} \end{matrix} B \rightrightarrows i \tag{1.8}$$

$$\begin{array}{ccc}
 f \hookrightarrow A & \xrightarrow{g} & B \rightrightarrows h \\
 & \searrow k & \downarrow i \\
 & & C \rightrightarrows j
 \end{array} \tag{1.9}$$

$$\begin{array}{ccc}
 f \hookrightarrow A & \xrightarrow{g} & B \rightrightarrows h \\
 & \searrow k & \downarrow i \\
 & & C \rightrightarrows j \\
 & \nearrow l &
 \end{array} \tag{1.10}$$

1.3 Opposite categories and duality

We will see a few ways of constructing new categories throughout those notes, but the first, and most important of those is that of the opposite category.

Definition 1.3.1. Let $\mathcal{C}$ be a category. Its opposite category, $\mathcal{C}^{\text{op}}$ is defined as follows.

- Its objects are the same as those of $\mathcal{C}$,
- For X, Y two objects of $\mathcal{C}^{\text{op}}$, define

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X).$$

For clarity, given a morphism $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, we will often write $f^{\text{op}}: Y \rightarrow X$ to designate the **same** morphism, but seen in $\mathcal{C}^{\text{op}}$, i.e. seen as an element of $\text{Hom}_{\mathcal{C}^{\text{op}}}(Y, X)$.

- for X an object of $\mathcal{C}^{\text{op}}$, define $\text{Id}_X^{\mathcal{C}^{\text{op}}} = \text{Id}_X^{\mathcal{C}} \in \text{Hom}_{\mathcal{C}}(X, X) = \text{Hom}_{\mathcal{C}^{\text{op}}}(X, X)$.
- for $f^{\text{op}}: X \rightarrow Y$ and $g^{\text{op}}: Y \rightarrow Z$ two composable morphisms in $\mathcal{C}^{\text{op}}$, define their composition $g^{\text{op}} \circ^{\text{op}} f^{\text{op}} = (f \circ g)^{\text{op}} \in \text{Hom}_{\mathcal{C}}(Z, X) = \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Z)$.

We will soon get used to arguments "by duality". The key idea here is that certain properties of a category (or in a category) come in pair, with one half being true in a category means that the other half is true in the opposite category. Let us illustrate this phenomenon with the notion of monomorphisms and epimorphisms.

Definition 1.3.2. Let $\mathcal{C}$ be a category, and $f: X \rightarrow Y$ be a morphism in $\mathcal{C}$.

- We say that f is a **monomorphism** if and only if, for any two morphisms $h_1, h_2: W \rightarrow X$, we have

$$f \circ h_1 = f \circ h_2 \Leftrightarrow h_1 = h_2$$

- We say that f is an **epimorphism** if, for any two morphisms $g_1, g_2: Y \rightarrow Z$, we have

$$g_1 \circ f = g_2 \circ f \Leftrightarrow g_1 = g_2$$

- We say that f is an **isomorphism** if and only if there exists $g: Y \rightarrow X$ such that

$$f \circ g = \text{Id}_Y \quad g \circ f = \text{Id}_X$$

In that case, we will often write f^{-1} instead of g . Furthermore, we will often write $f: X \simeq Y$ instead of $f: X \rightarrow Y$.

Exercise 6. Let $\mathcal{C}$ be a category, show that $(\mathcal{C}^{\text{op}})^{\text{op}}$ and $\mathcal{C}$ are the same category.

Exercise 7. Let $\mathcal{C}$ be a category, and $f: X \rightarrow Y$ be a morphism in $\mathcal{C}$. Show that

1. f is a monomorphism if and only if f^{op} is an epimorphism in $\mathcal{C}^{\text{op}}$.
2. f is an epimorphism if and only if f^{op} is a monomorphism in $\mathcal{C}^{\text{op}}$.
3. f is an isomorphism if and only if f^{op} is an isomorphism in $\mathcal{C}^{\text{op}}$.
4. If f is an isomorphism, then f is a monomorphism and an epimorphism.

Exercise 8. Let $f: X \rightarrow Y$ be a morphism in $\mathbf{Set}$. When is f a monomorphism ? an epimorphism ? an isomorphism ? Same question, for the categories of examples 7 and 8 in Example 1.1.4. Can a morphism be both a monomorphism and an epimorphism but not an isomorphism ?

1.4 Subcategories

Another fundamental construction one can do with a category is consider categories that are included in it. Defining new categories this way will allow to skip the repetitive checking of axioms.

Definition 1.4.1. Let $\mathcal{C}$ be a category. A subcategory of $\mathcal{C}$, denoted $\mathcal{D} \subset \mathcal{C}$, is a category such that

- each object of $\mathcal{D}$ is an object of $\mathcal{C}$,
- For X, Y objects of $\mathcal{D}$, $\text{Hom}_{\mathcal{D}}(X, Y) \subset \text{Hom}_{\mathcal{C}}(X, Y)$,
- for all X objects of $\mathcal{D}$, $\text{Id}_X^{\mathcal{D}} = \text{Id}_X^{\mathcal{C}} \in \text{Hom}_{\mathcal{D}}(X, X)$.
- for $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ two morphisms in $\mathcal{D}$, we have $g \circ^{\mathcal{D}} f = g \circ^{\mathcal{C}} f \in \text{Hom}_{\mathcal{D}}(X, Z)$.

The notion of subcategories turns out to be a very convenient way to define categories. Indeed, the axioms of a category are guaranteed for a subcategory.

Exercise 9. Let $\mathcal{C}$ be a category, and define the objects and morphisms of a category $\mathcal{D}$ such that

- each object of $\mathcal{D}$ is an object of $\mathcal{C}$,
- for all X, Y objects of $\mathcal{D}$, $\text{Hom}_{\mathcal{D}}(X, Y) \subset \text{Hom}_{\mathcal{C}}(X, Y)$,
- for all X object of $\mathcal{D}$, $\text{Id}_X^{\mathcal{C}} \in \text{Hom}_{\mathcal{D}}(X, X)$,
- for all $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ in $\mathcal{D}$, $g \circ^{\mathcal{C}} f \in \text{Hom}_{\mathcal{D}}(X, Z)$

Show that $\mathcal{D}$ admits the structure of a category by restricting the identities and composition from $\mathcal{C}$, and that it makes $\mathcal{D}$ into a subcategory of $\mathcal{C}$.

The above exercise shows that, in order to define a subcategory, it is enough to define its objects and morphisms. The definition of identity and compositions, and checking the axioms for a category are then replaced by two simple checks. We mention a particularly convenient case.

Definition 1.4.2. Let $\mathcal{C}$ be a category. A full subcategory of $\mathcal{C}$ is a subcategory $\mathcal{D}$ such that for all X, Y objects of $\mathcal{D}$, $\text{Hom}_{\mathcal{D}}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)$.

Remark 1.4.3. Defining a full subcategory is even simpler. It is enough to specify the class of objects, and the checks that it forms a subcategory are automatic.

1.5 About sets, classes, and universes

Those notes on category theory are based on (a very slightly enriched version of) set theory. This section is only here to explain how this (surprisingly non-standard) approach leads to a treatment of the subject that may differ in parts to more classical approaches (For example, the approach in [Mac71]). The discussion in this section will not be used anywhere else in those notes, and thus can be safely skipped.

Throughout the text we will work with classes and sets. Here are the constructions we allow ourselves to work with.

- for sets, we will use the usual constructions of set theory, though we will not ever need any **specific** implementation. Specifically :
 - We will consider elements of sets, denoted $x \in X$
 - We may compare elements in sets, given $x, y \in X$, we may write $x = y$ or $x \neq y$.
 - We will assume that arbitrary (set-indexed) disjoint unions of sets $\coprod_{a \in A} X_a$ and arbitrary products of sets $\prod_{b \in B} Y_b$ are defined in an unambiguous way (though, the **specific** implementation of such a definition is irrelevant).
 - We will consider maps of sets $f: X \rightarrow Y$.
 - We will assume that the set of all such maps is defined in an unambiguous way, $\text{Fct}(X, Y)$, together with an evaluation map

$$\begin{aligned} \text{ev}: \text{Fct}(X, Y) \times X &\rightarrow Y \\ (f, x) &\mapsto f(x) \end{aligned}$$

- Notably, we will not ever need to use claims to the effect of "any element of a set is itself a set".
- Unless two sets X and Y happen to be elements of some common set Z under consideration, we will abstain from making claims such as $X = Y$ or $X \neq Y$. Though we will always be interested in whether or not a bijection between two sets exists.
- A caveat of the above comment is for the case of subcategories. Though, the equality (or inclusion) between sets of morphisms should be understood as definitional in that case.
- For classes :

- We will assume that there exists a class of **all** sets. More generally, we will allow ourselves to call **any** collection of objects a class. The set theorist worried about paradoxes here may assume that by "objects", we really mean "sets", though this would force us back into the philosophy of "everything is a set". An easier (but less precise) solution is to abstain from considering classes of classes, and more generally, we will abstain from quantifying on classes.
- We will allow ourselves to interpret any set as a class.
- We will consider objects in a class, and may even abuse the notation $x \in \mathcal{C}$.
- Given two objects in a class, x and y we will consider the statements $x = y$ and $x \neq y$ as meaningless *a priori*. More generally, as a philosophy, we will consider that a statement of the form $x = y$ should only be formulated when x and y are two elements in a common set.
- We may consider set-indexed products or disjoint unions of classes. (In the process of proving Proposition 2.3.5, we will need to consider choice functions on classes, which implies in particular that we may consider a class-indexed product of classes. This will be the only such occurrence, and no other result will rely on this proposition.)
- We will consider function between classes, in a restricted meaning. Given two classes $\mathcal{C}$ and $\mathcal{D}$, we will consider maps $f: \mathcal{C} \rightarrow \mathcal{D}$. Those should be interpreted as some construction of an element $f(x)$ in $\mathcal{D}$ for any element x of $\mathcal{C}$. Note that in the process of defining such maps, the symbol "=" will appear. This, of course, should be interpreted as definitional equality.
- We may **not** assume that there exists a class of maps between two classes. *a fortiori*, we will not consider statements of the form $f = g$ or $f \neq g$ for two maps between classes.

The above technology should feel tame to anybody used to set-theory. Essentially, classes are a notion of meta-set, which encompasses the collection of all sets. Essentially, the only thing we can do with such a class is consider its elements (though not compare them).

Contrast this approach with the three main approaches, by order of required technology.

- One may work strictly with (ZFC) set-theory. This approach sacrifices a lot of the expressive power of category-theory. Indeed, without a "thing" containing all sets, we cannot define the category of sets. As a consequence, we actually lose access to **all** categories built on sets without restriction : the categories of vector spaces, of topological spaces, of graphs, etc. Instead, one need to systematically use tricks to study objects of interest. For example one may consider the set $\{\mathbb{R}^n \mid n \in \mathbb{N}\}$ instead of the collection of all finite dimensional vector spaces, which does not form a set. Working in this context leads to the theory of **small categories**.

- The historical approach of MacLane starts with the (unspoken) assumption that a theory of classes exist, and that one can do anything with classes that can be done with sets (see Chapter I.1 in [Mac71]). The crucial difference with our point of view is the assumption that one can meaningfully compare elements in a class. From the point of view of MacLane, the theory we develop in those notes is a theory of **locally-small categories**.
- A more modern approach is to forget entirely about classes, and instead consider universes. Those are (extra-large) sets, whose existence are added as extra axioms to (ZFC) set-theory. There is a large variety of possible axioms, lying far outside the scope of this class, but we will outline the two main versions of this point of view.
 - One may ask that there exist some uncountable set, U , called a universe, such that U itself is a model of ZFC set-theory. Elements of U will then be called **small** sets (in this context, the "everything is a set" philosophy becomes somewhat mandatory). To expand on this assumption, the fact that U is a model for ZFC means that, given any set $X \in U$, the set of parts of X , $P(X)$ is also an element of U . Even more, any small-set-indexed union of elements of U is still an element of U . This allows to define a version of the category of sets : the category of all **small** sets. This approach can be seen as a way to formalize the aforementioned approach of MacLane
 - One may make the stronger assumption that for **any** set, X , there exists some universe U such that $X \in U$. We will say that X is U -small in that case. This leads to the existence of a **hierarchy** of universes, where each universe is an element in the next. To illustrate the power of this point of view, observe the following. First, as we saw, we can now define a category of sets (by considering only the U -small sets, i.e. those that are sitting inside a chosen universe U). But now, we may want to consider the category of **all** categories. In particular, this must contain the category of sets. This is now possible. Indeed, by assumption there exists some other universe V such that $U \in V$. Hence, we may simply define the category of all V -small categories. But we may even go further. Since $V \in W$ for some universe W , we may consider the category of all W -small categories. It contains the category of all V -small categories. Here is what one should take away from the above. First, "anything can be assumed to be a set, up to passing to a larger universe". But also this shows that one can not **ever** forget completely about universes, without bringing back paradoxes. Indeed, if one forgets about the different kinds of smallness, one may mistakenly think that we have just constructed a category that contains itself as an object. This is the reason why this is not the approach taken in these notes. Working **rigorously** with universes forces one to keep track of universes constantly. And being not rigorous leads to downright nonsensical statements. See

Proposition [4.3.12](#), and the following remark.

Chapter 2

Functors and natural transformations

In the previous chapter, we have defined categories, and given several examples, but we considered each of those separately. We now want to be able to relate categories to one another. We thus introduce **functors**, which should be interpreted as morphisms between categories. In fact, at least for small categories, this intuition is entirely accurate, as we will see in Definition 2.1.3.

2.1 Functors

Definition 2.1.1. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between two categories $\mathcal{C}$ and $\mathcal{D}$ is the data of

- For each object, X , of $\mathcal{C}$, an object $F(X)$ of $\mathcal{D}$,
- For each map $f: X \rightarrow Y$ in $\mathcal{C}$, a map $F(f): F(X) \rightarrow F(Y)$ in $\mathcal{D}$.

such that

- For all objects X of $\mathcal{C}$, $F(\text{Id}_X) = \text{Id}_{F(X)} \in \text{Hom}_{\mathcal{D}}(F(X), F(X))$,
- For all pairs of composable arrows in $\mathcal{C}$, $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, we have

$$F(g \circ f) = F(g) \circ F(f) \in \text{Hom}_{\mathcal{D}}(F(X), F(Z))$$

Example 2.1.2. • Define a functor $\text{Forget}: \underline{\text{Grp}} \rightarrow \underline{\text{Set}}$, sending a group $(G, \cdot)$ to its underlying set $\text{Forget}(G, \cdot) = \underline{G}$, and sending a group homomorphism to the underlying map of set.

- More generally, given any kind of algebraic structure on a set, i.e. a group structure, a ring structure, a vector space structure, etc. We may define a **forgetful functor** from the category of such objects to $\underline{\text{Set}}$, which forgets the algebraic structure.

- Let k be a field, consider the functor from $\mathbf{k}\text{-Vect}$ to $\mathbf{Grp}$, sending the vector space $(V, +, \cdot)$ to the group $(V, +)$. This is another kind of forgetful functor. This time, it forgets only parts of the algebraic structure.
- Let $\mathbf{Ab}$ be the category of abelian groups and group homomorphisms. Consider the functor $\mathbf{Ab} \rightarrow \mathbf{Grp}$ sending an abelian group $(A, +)$ to itself. This is in fact the inclusion of a full subcategory, but can also be thought of as a forgetful functor. It "forgets" about the fact that A was an abelian group.
- For an example of another kind, let $\mathcal{C}$ be some category, and X some object of $\mathcal{C}$. Define the functor

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, -): \mathcal{C} &\rightarrow \mathbf{Set} \\ Y &\mapsto \text{Hom}_{\mathcal{C}}(X, Y) \\ f: Y \rightarrow Z &\mapsto \begin{cases} \text{Hom}_{\mathcal{C}}(X, f): \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{C}}(X, Z) \\ g &\mapsto f \circ g \end{cases} \end{aligned}$$

Exercise 10. Check that the above is indeed a functor.

- Let k be a field, and consider the following.

$$\begin{aligned} F: \mathbf{k}\text{-Mat} &\rightarrow \mathbf{k}\text{-Vect} \\ n &\mapsto k^n \\ M \in \text{Hom}_{\mathbf{k}\text{-Mat}}(m, n) &\mapsto \begin{cases} k^m &\rightarrow k^n \\ v &\mapsto Mv \end{cases} \end{aligned}$$

What we know of the theory of matrices and linear algebra tell us that this is indeed a functor.

Exercise 11. Let $\underline{P}$ and $\underline{Q}$ be two pre-ordered sets seen as categories. What is a functor $F: \underline{P} \rightarrow \underline{Q}$? Same question for two monoids seen as categories.

Exercise 12. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

- Show that F sends isomorphisms in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$.
- Find an example of $F: \mathcal{C} \rightarrow \mathcal{D}$, $f: X \rightarrow Y$ a morphism in $\mathcal{C}$ such that f is a monomorphism but $F(f)$ is not a monomorphism. Same question for epimorphisms.
- We say that a monomorphism, $f: X \rightarrow Y$ is split if there exists $g: Y \rightarrow X$ such that $g \circ f = \text{Id}_X$.
 - show that if $f: X \rightarrow Y$ is any morphism such that there exists $g: Y \rightarrow X$ such that $g \circ f = \text{Id}_X$, then f is a monomorphism.
 - Show that if f is a split monomorphism, then $F(f)$ is a split monomorphism.

2.1.1 The category of small categories

Definition 2.1.3. Let $\underline{\mathbf{Cat}}$ be the category where

- its objects are small categories
- given $\mathcal{C}$ and $\mathcal{D}$ two small categories, $\mathrm{Hom}_{\underline{\mathbf{Cat}}}(\mathcal{C}, \mathcal{D})$ is the set of functors from $\mathcal{C}$ to $\mathcal{D}$,
- given a small category $\mathcal{C}$, $\mathrm{Id}_{\mathcal{C}}$ is the functor sending every object of $\mathcal{C}$ to itself, and every map in $\mathcal{C}$ to itself.
- Given $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{E}$ the composition is defined as the functor $G \circ F: \mathcal{C} \rightarrow \mathcal{E}$ sending an object X of $\mathcal{C}$ to the object $G(F(X))$ in $\mathcal{E}$, and sending $f: X \rightarrow Y$ to $G(F(f)): G(F(X)) \rightarrow G(F(Y))$.

Remark 2.1.4. It is worth pondering the implications of the above definition. When studying a particular category, say, the category of groups for example, one may think of it as studying group theory, in a certain sense. Thus, studying the category of all (small) categories, allows one to study category theory, *in the language of category theory*. One could say that studying $\underline{\mathbf{Cat}}$ is akin to studying the meta-theory of all theories. Certainly, the previous statement is, too ambitious (and too naïve) to be meaningful. However, it is true, and somewhat of a common technique, that studying $\underline{\mathbf{Cat}}$ (and further generalizations of it) can - and does - give insights into what kind of results can be obtained when working with category theory. Indeed, this line of thinking will lead to fairly useful meta-theorems, that can then be applied to specific categories. See for example Proposition 2.3.5, at the end of this chapter, and how it applies to the concrete example in the second part of Example 2.3.2.

Definition 2.1.5. A functor between small categories $F: \mathcal{C} \rightarrow \mathcal{D}$ is an **isomorphism of categories** if it is an isomorphism in $\underline{\mathbf{Cat}}$. i.e. if there exists a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $F \circ G = \mathrm{Id}_{\mathcal{D}}$ and $G \circ F = \mathrm{Id}_{\mathcal{C}}$.

Remark 2.1.6. The above definition no longer makes sense for general categories, since it relies on checking equality between functors. But this can only be checked if equality between objects are meaningful, which is not the case for objects in a non-small category.

Example 2.1.7. Let $\mathcal{C} \subset \underline{\mathbf{Set}}$ be the full subcategory containing the objects $\emptyset$ and $\{\emptyset\}$. Let P be the poset $\{0 < 1\}$, and let $\mathcal{D}$ be the category corresponding to the following graph

$$\mathrm{Id}_A \hookrightarrow A \xrightarrow{f} B \rightrightarrows \mathrm{Id}_B$$

Then, those three categories are isomorphic. The isomorphism from $\mathcal{C}$ to P is

given by:

$$\begin{aligned}\mathcal{C} &\rightarrow \underline{P} \\ \emptyset &\mapsto 0 \\ \{\emptyset\} &\mapsto 1 \\ \emptyset \rightarrow \{\emptyset\} &\mapsto 0 \rightarrow 1\end{aligned}$$

The isomorphism from $\underline{P}$ to $\mathcal{D}$ is obtained by sending 0 to A and 1 to B , and the isomorphism from $\mathcal{D}$ to $\mathcal{C}$ is obtained by sending A to $\emptyset$ and B to $\{\emptyset\}$.

Exercise 13. List as many non-isomorphic categories as you can with at most 5 morphisms including identities. (Hint : there are way more than you think).

Exercise 14. Consider the following

$$\begin{aligned}(-)^{\text{op}}: \underline{\text{Cat}} &\rightarrow \underline{\text{Cat}} \\ \mathcal{C} &\mapsto \mathcal{C}^{\text{op}} \\ F: \mathcal{C} \rightarrow \mathcal{D} &\mapsto \begin{cases} F^{\text{op}}: \mathcal{C}^{\text{op}} &\rightarrow \mathcal{D}^{\text{op}} \\ X &\mapsto F(X) \\ f^{\text{op}}: Y \rightarrow X &\mapsto F(f)^{\text{op}}: F(Y) \rightarrow F(X) \end{cases}\end{aligned}$$

Show that the above defines a functor.

2.1.2 Contravariant functors

We will also need to consider a slight variation on the notion of functors : **Contravariant functors**, which reverse the direction of arrows.

Definition 2.1.8. A **contravariant functor** $F: \mathcal{C} \rightarrow \mathcal{D}$ is a functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$. Whenever ambiguity is possible, we will refer to functors in the usual sense as **covariant** functors.

Exercise 15. Using Exercise 14, together with the observation that $(\mathcal{C}^{\text{op}})^{\text{op}}$ is $\mathcal{C}$, show that a contravariant functor is equivalently defined as a functor $F: \mathcal{C} \rightarrow \mathcal{D}^{\text{op}}$.

The main (and maybe only) reason to care about contravariant functors is the following fundamental example. Compare with the second to last example of Example 2.1.2.

Example 2.1.9. Let $\mathcal{C}$ be some category and X some object of $\mathcal{C}$. Consider the (covariant) functor

$$\begin{aligned}\text{Hom}_{\mathcal{C}}(-, X): \mathcal{C}^{\text{op}} &\rightarrow \underline{\text{Set}} \\ Y &\mapsto \text{Hom}_{\mathcal{C}}(Y, X) \\ f: Y \rightarrow Z &\mapsto \begin{cases} \text{Hom}_{\mathcal{C}}(f, X): \text{Hom}_{\mathcal{C}}(Z, X) &\rightarrow \text{Hom}_{\mathcal{C}}(Y, X) \\ g &\mapsto g \circ f. \end{cases}\end{aligned}$$

It can equivalently be described as a *contravariant* functor, $\mathcal{C} \rightarrow \underline{\text{Set}}$.

Exercise 16. Find a covariant functor $F: \underline{\text{Set}} \rightarrow \underline{\text{Set}}$ and a contravariant functor $G: \underline{\text{Set}} \rightarrow \underline{\text{Set}}$ such that for any set X , $F(X) = G(X) = \mathcal{P}(X)$ is the set of parts of X .

2.1.3 Properties of functors

To state the following property, it will be convenient to make an observation. Given a functor $F: \mathcal{C} \rightarrow \mathcal{D}$, and two objects X, Y of $\mathcal{C}$, F sends morphisms in $\text{Hom}_{\mathcal{C}}(X, Y)$ to morphisms in $\text{Hom}_{\mathcal{D}}(F(X), F(Y))$. In particular, it will often be convenient to think of F as giving a collection of set-theoretic maps

$$\begin{aligned} F: \text{Hom}_{\mathcal{C}}(X, Y) &\rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ f &\mapsto F(f) \end{aligned} \quad (2.1)$$

Definition 2.1.10. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

- It is said to be **full** if, for all X, Y in $\mathcal{C}$, the map (2.1) is surjective,
- It is said to be **faithful** if, for all X, Y in $\mathcal{C}$, the map (2.1) is injective,
- It is said to be **fully faithful** if, for all X, Y in $\mathcal{C}$, the map (2.1) is bijective.

Exercise 17. Let $\mathcal{D}$ be a category, and $\mathcal{C} \subset \mathcal{D}$ a subcategory.

- Make explicit the obvious functor $F: \mathcal{C} \rightarrow \mathcal{D}$.
- Show that F must be faithful.
- Show that F is fully-faithful if and only if $\mathcal{C}$ is a full subcategory of $\mathcal{D}$.

Definition 2.1.11. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. It is said to be **essentially surjective** if, for all object of $\mathcal{D}$, Y , there exists some object of $\mathcal{C}$, X , and an isomorphism in $\mathcal{D}$ between $F(X)$ and Y .

Exercise 18. Let k be a field, and let $\underline{k\text{-Vect}}_{\text{fin}} \subset \underline{k\text{-Vect}}$ the full subcategory of finite dimensional vector spaces. Consider the functor $F: \underline{k\text{-Mat}} \rightarrow \underline{k\text{-Vect}}$ of Example 2.1.2.

- Show that F factor through some functor $F': \underline{k\text{-Mat}} \rightarrow \underline{k\text{-Vect}}_{\text{fin}}$.
- Show that F' is fully faithful.
- Show that F' is essentially surjective.

2.2 Natural transformations

We have seen that functors are a way of relating categories to one another, but we will also want to compare functors, and relate them to one another. In general, checking whether two functors are equal will be meaningless, but we may instead consider morphisms between them. Those will be called natural transformations. The ill-posed question of equality between functors will then be replaced by the existence (or non-existence) of isomorphisms between them (see Definition 2.2.8).

Definition 2.2.1. Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be two functors. A **natural transformation** α from F to G , written $\alpha: F \rightarrow G$ (or sometimes $\alpha: F \Rightarrow G$), is a collection of maps (α_X) , where X ranges over all objects of $\mathcal{C}$, where $\alpha_X: F(X) \rightarrow G(X)$ is a map in $\mathcal{D}$, such that, for all maps $f: X \rightarrow Y$ in $\mathcal{C}$, the following diagram is commutative :

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ F(f) \downarrow & & \downarrow G(f) \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array} \quad (2.2)$$

Example 2.2.2. Let Z be a set, $F: \underline{\text{Set}} \rightarrow \underline{\text{Set}}$ the identity functor, and consider the functor $G: \underline{\text{Set}} \rightarrow \underline{\text{Set}}$ defined as follows

$$\begin{aligned} G: \underline{\text{Set}} &\rightarrow \underline{\text{Set}} \\ X &\mapsto Z \times X \\ f: X \rightarrow Y &\mapsto \begin{cases} Z \times X & \rightarrow Z \times Y \\ (z, x) & \mapsto (z, f(x)) \end{cases} \end{aligned}$$

pick some element $z_0 \in Z$. And, for all X , define the map $\alpha_X: X \rightarrow Z \times X$ sending x to (z_0, x) . Let us show that the collection of maps (α_X) forms a natural transformation. Let $f: X \rightarrow Y$ be some map in $\underline{\text{Set}}$. We need to show that the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{\alpha_X} & Z \times X \\ f \downarrow & & \downarrow G(f) \\ Y & \xrightarrow{\alpha_Y} & Z \times Y \end{array}$$

Now, let $x \in X$ be some element. We compute, $G(f) \circ \alpha_X(x) = G(f)(\alpha_X(x)) = G(f)(z_0, x) = (z_0, f(x))$. On the other hand, we have $\alpha_Y \circ f(x) = \alpha_Y(f(x)) = (z_0, f(x))$. Hence, $G(f) \circ \alpha_X = \alpha_Y \circ f$, and the diagram is indeed commutative. This shows that $\alpha: F \rightarrow G$ is a natural transformation.

Now, for any set X consider the map $\text{pr}_X: Z \times X \rightarrow X$. Let us show that those maps give a natural transformation $\text{pr}: G \rightarrow F$. Once again, we must check that the following diagram is commutative, for any map $f: X \rightarrow Y$ in $\mathcal{C}$.

$$\begin{array}{ccc} Z \times X & \xrightarrow{\text{pr}_X} & X \\ G(f) \downarrow & & \downarrow f \\ Z \times Y & \xrightarrow{\text{pr}_Y} & Y \end{array}$$

Given $(z, x) \in Z \times X$, we compute $f \circ \text{pr}_X(z, x) = f(x)$ and $\text{pr}_Y \circ G(f)(z, x) = \text{pr}_Y(z, f(x)) = f(x)$. Hence, the diagram is commutative, and we have a natural transformation $\text{pr}: G \rightarrow F$.

Remark 2.2.3. Natural transformations are named this way because they were defined to match the intuition behind "natural constructions" in mathematics. This idea of naturality is very common, and not always made precise. But, very often when some map seems natural, (or canonical), there actually exists some natural transformation behind it.

Exercise 19. Find occurrences of maps who are claimed to be "natural" in other lecture notes (not necessarily about category theory) and try to find the functors and natural transformations in question.

Exercise 20. Let P and P' be two pre-order. What is a natural transformation between two functors $F, G: \underline{P} \rightarrow \underline{P}'$? Same question for groups seen as one object categories.

Example 2.2.4. Let $f: X \rightarrow Y$ be a map in some category $\mathcal{C}$. Consider the functors $\text{Hom}_{\mathcal{C}}(X, -), \text{Hom}_{\mathcal{C}}(Y, -): \mathcal{C} \rightarrow \underline{\text{Set}}$, defined in Example 2.1.2. Given some object Z of $\mathcal{C}$, define a map:

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(f, -)_Z &= \text{Hom}_{\mathcal{C}}(f, Z): \text{Hom}_{\mathcal{C}}(Y, Z) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z) \\ g: Y \rightarrow Z &\mapsto g \circ f: X \rightarrow Z \end{aligned}$$

Let us show that the collection of maps $(\text{Hom}_{\mathcal{C}}(f, Z))$ forms a natural transformation from $\text{Hom}_{\mathcal{C}}(Y, -)$ to $\text{Hom}_{\mathcal{C}}(X, -)$. Let $h: Z \rightarrow W$ be a map in $\mathcal{C}$, we need to show that the following diagram is commutative

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(Y, Z) & \xrightarrow{\text{Hom}_{\mathcal{C}}(f, Z)} & \text{Hom}_{\mathcal{C}}(X, Z) \\ \text{Hom}_{\mathcal{C}}(Y, h) \downarrow & & \downarrow \text{Hom}_{\mathcal{C}}(X, h) \\ \text{Hom}_{\mathcal{C}}(Y, W) & \xrightarrow{\text{Hom}_{\mathcal{C}}(f, W)} & \text{Hom}_{\mathcal{C}}(X, W) \end{array} \quad (2.3)$$

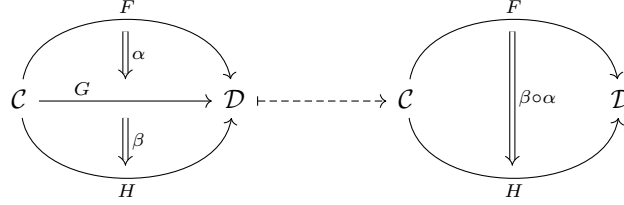
Before proving commutativity, take the time to parse the diagram correctly. The horizontal arrows are the maps that will form a natural transformation, while the vertical arrows are obtained by applying the functors $\text{Hom}_{\mathcal{C}}(Y, -)$ and $\text{Hom}_{\mathcal{C}}(Z, -)$ to the morphism $h: Z \rightarrow W$. Now, let $g: Y \rightarrow Z$ be a morphism. Applying the composition of the top and right arrows to g gives the map $h \circ (g \circ f): X \rightarrow W$. Going through the left and bottom arrows gives the map $(h \circ g) \circ f$. Associativity of the composition gives the equality $h \circ (g \circ f) = (h \circ g) \circ f$. Hence, we deduce that the square is indeed commutative. This shows that $\text{Hom}_{\mathcal{C}}(f, -): \text{Hom}_{\mathcal{C}}(Y, -) \rightarrow \text{Hom}_{\mathcal{C}}(X, -)$ is a natural transformation.

Exercise 21. Let $h: Z \rightarrow W$ be a map in some category $\mathcal{C}$. Using the commutativity of the diagram (2.3), show that $\text{Hom}_{\mathcal{C}}(-, h): \text{Hom}_{\mathcal{C}}(-, Z) \rightarrow \text{Hom}_{\mathcal{C}}(-, W)$ is a natural transformation between the contravariant functors of Example 2.1.9.

2.2.1 Composing natural transformations

Definition 2.2.5. Let $F, G, H: \mathcal{C} \rightarrow \mathcal{D}$ be three functors, and $\alpha: F \rightarrow G$ and $\beta: G \rightarrow H$ be two natural transformations. Let $\beta \circ \alpha: F \rightarrow H$ stand for the natural transformation defined via $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X: F(X) \rightarrow H(X)$, for all

objects X of $\mathcal{C}$. the natural transformation $\beta \circ \alpha$ is called the **vertical composition** of β with α . The following diagram illustrates why.



Exercise 22. Show that $\beta \circ \alpha$ is indeed a natural transformation, and that the composition of natural transformations is associative.

Definition 2.2.6. Let $\mathcal{C}$ be a small category and $\mathcal{D}$ any category. Define the category of functors and natural transformations from $\mathcal{C}$ to $\mathcal{D}$, $\text{Fun}(\mathcal{C}, \mathcal{D})$, as follows. (We will often just call it the category of functors from $\mathcal{C}$ to $\mathcal{D}$.)

- Its objects are functors $F: \mathcal{C} \rightarrow \mathcal{D}$,
- Given two functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$, define the set of morphism,

$$\text{Hom}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(F, G) \subset \prod_{X \in \text{obj}(\mathcal{C})} \text{Hom}_{\mathcal{D}}(F(X), G(X)),$$

as the set of natural transformations from F to G .

- given a functor $F: \mathcal{C} \rightarrow \mathcal{D}$, the identity of F is the natural transformation $\text{Id}_F: F \rightarrow F$ defined, for all X via $(\text{Id}_F)_X = \text{Id}_{F(X)}: F(X) \rightarrow F(X)$.
- The composition is the one defined in Definition 2.2.5.

Remark 2.2.7. In the above definition, we asked that $\mathcal{C}$ was a *small* category. This is needed so that the collection of all natural transformations between two functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ may be a *set*. Indeed, if $\mathcal{C}$ is small, then the set of natural transformations is defined as a subset of a product of sets, hence is certainly a set. On the other hand, if $\mathcal{C}$ was not a small category, then the collection of all natural transformations between two given functors would be a class in general, and thus $\text{Fun}(\mathcal{C}, \mathcal{D})$ would fail to be a category. Note, however, that even if $\mathcal{C}$ is small, $\text{Fun}(\mathcal{C}, \mathcal{D})$ will only remain a small category if $\mathcal{D}$ is *also* a small category.

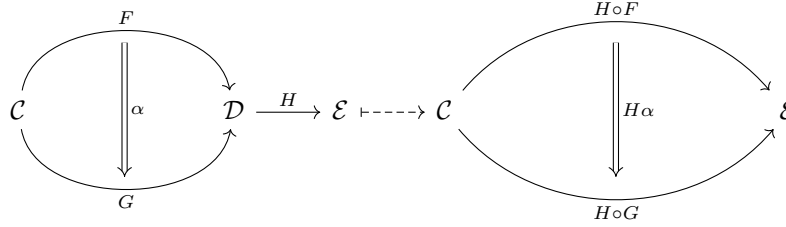
Definition 2.2.8. A natural transformation $\alpha: F \rightarrow G$ between two functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$ is said to be a **natural isomorphism** if, for all objects X in $\mathcal{C}$, $\alpha_X: F(X) \rightarrow G(X)$ is an isomorphism.

Exercise 23. Let $\alpha: F \rightarrow G$ be a natural transformation between two functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$.

- Prove that α is a natural isomorphism if and only if there exists some natural transformation $\beta: G \rightarrow F$ such that $\alpha \circ \beta = \text{Id}_G$ and $\beta \circ \alpha = \text{Id}_F$.
- Deduce that, if $\mathcal{C}$ is a small category, isomorphisms in $\text{Fun}(\mathcal{C}, \mathcal{D})$ are exactly natural isomorphisms.

2.2.2 Horizontal composition of natural transformation

Definition 2.2.9. Assume that we are given three categories, $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$, three functors, $F, G: \mathcal{C} \rightarrow \mathcal{D}$ and $H: \mathcal{D} \rightarrow \mathcal{E}$, and a natural transformation, $\alpha: F \rightarrow G$. Then, we can define a natural transformation $H\alpha: H \circ F \rightarrow H \circ G$ as follows. For X an object of $\mathcal{C}$, define $(H\alpha)_X = H(\alpha_X): H(F(X)) \rightarrow H(G(X))$. The situation is summarized in the following diagram



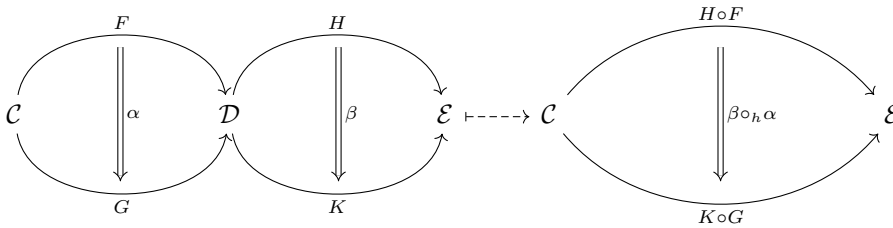
Let us prove that $H\alpha$ is indeed a natural transformation. Let $f: X \rightarrow Y$ be a map in $\mathcal{C}$. Consider the diagram

$$\begin{array}{ccc} H \circ F(X) & \xrightarrow{H(\alpha_X)} & H \circ G(X) \\ H \circ F(f) \downarrow & & \downarrow H \circ G(f) \\ H \circ F(Y) & \xrightarrow{H(\alpha_Y)} & H \circ G(Y) \end{array}$$

The top right composition is equal to $H(G(f)) \circ H(\alpha_X) = H(G(f) \circ \alpha_X)$, by functoriality of H . By the same argument, the left-bottom composition gives $H(\alpha_Y) \circ H(F(f)) = H(\alpha_Y \circ F(f))$. Now, since α is a natural transformation, we must have $\alpha_Y \circ F(f) = G(f) \circ \alpha_X$, hence the equality still holds when applying H . This proves that the square is commutative, hence that $H\alpha$ is a natural transformation.

Remark 2.2.10. Of course, the dual situation is also true. Given a natural transformation $\beta: H \rightarrow K$ between two functors $H, K: \mathcal{D} \rightarrow \mathcal{E}$, and a functor $F: \mathcal{C} \rightarrow \mathcal{D}$, we can define the composition $\beta F: H \circ F \rightarrow K \circ F$. The proof is the same as above.

Proposition 2.2.11. Assume that we are given three categories $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$, four functors, $F, G: \mathcal{C} \rightarrow \mathcal{D}$ and $H, K: \mathcal{D} \rightarrow \mathcal{E}$, and two natural transformations $\alpha: F \rightarrow G$ and $\beta: G \rightarrow H$. Then, there exists a well defined natural transformation called the **horizontal** composition $\beta \circ_h \alpha: H \circ F \rightarrow K \circ G$. The situation can be summarized as follows.



Proof. We will provide two possible definitions for $\beta \circ_h \alpha$ and show that they coincide. First, one can consider the natural transformation $\beta F: H \circ F \rightarrow K \circ F$, and $K\alpha: K \circ F \rightarrow K \circ G$, and define $\beta \circ_h \alpha = K\alpha \circ \beta F: H \circ F \rightarrow K \circ G$. This gives us a natural transformation of the desired form.

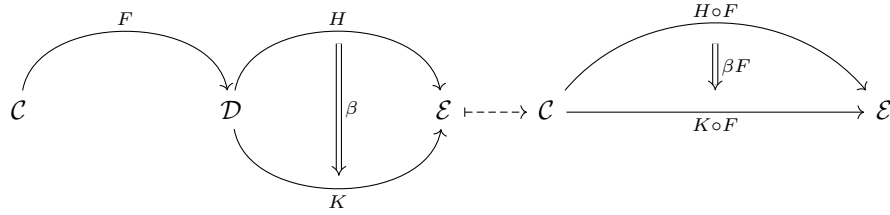
On the other hand, we could have considered first $H\alpha: H \circ F \rightarrow H \circ G$, then $\beta G: H \circ G \rightarrow K \circ G$, and then defined $\beta \circ_h \alpha = \beta G \circ H\alpha$. Let us now prove that both definitions coincide. Let X be some object in $\mathcal{C}$, and consider the following diagram.

$$\begin{array}{ccc} H(F(X)) & \xrightarrow{\beta_{F(X)}} & K(F(X)) \\ H(\alpha_X) \downarrow & & \downarrow K(\alpha_X) \\ H(G(X)) & \xrightarrow{\beta_{G(X)}} & K(G(X)) \end{array}$$

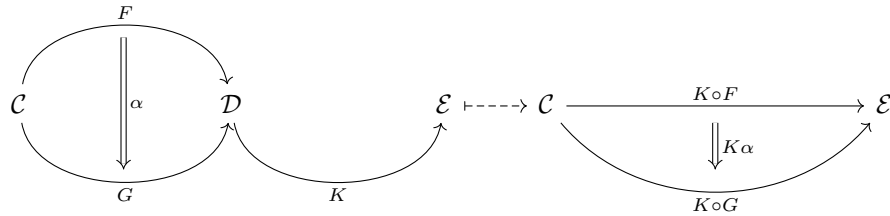
Observe that the composition of the top and right arrows is exactly the natural transformation $K\alpha \circ \beta F$ evaluated at X , while the composition of the left and bottom arrows gives the natural transformation $\beta G \circ H\alpha$ evaluated at X . On the other hand, this square must commute since β is a natural transformation, and it corresponds exactly to the square (2.2) applied to the natural transformation β and the map $f = \alpha_X: F(X) \rightarrow G(X)$. This means that the two way of defining $\beta \circ_h \alpha$ coincide.

We can summarize this construction diagrammatically.

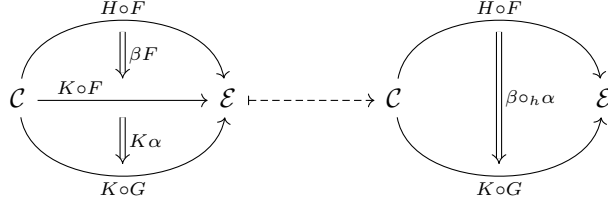
First consider βF :



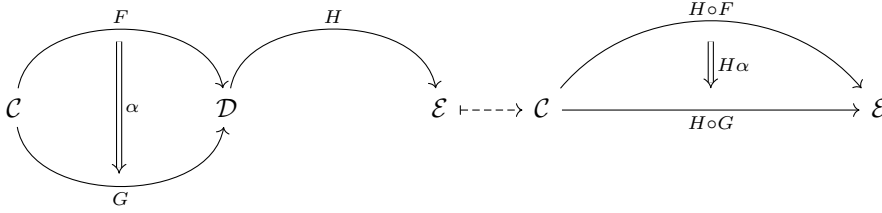
Then, define $K\alpha$:



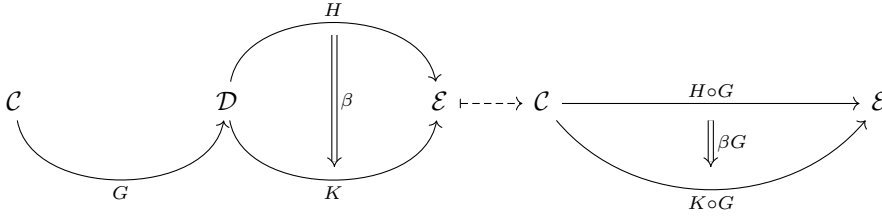
Then compose the two vertically to obtain $\beta \circ_h \alpha$.



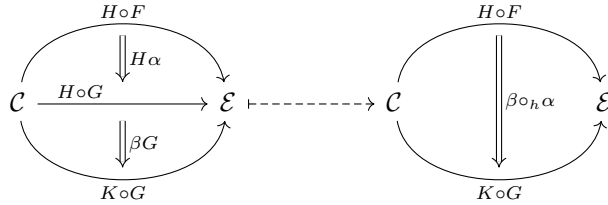
Equivalently, one may first consider $H\alpha$



Then, define βG :



and define $\beta \circ_h \alpha$ as the vertical composition



□

Remark 2.2.12. The above notions of horizontal and vertical compositions naturally leads one to consider 2-categories. Those are a generalization of categories, whose definition can be summarized as follows. A 2-category $\mathcal{C}$ has a collection of objects, and for any two objects X and Y , $\text{Hom}_{\mathcal{C}}(X, Y)$ should itself be a category (in the usual sense). For X, Y and Z objects in $\mathcal{C}$, there should be a composition functor

$$\text{Hom}_{\mathcal{C}}(Y, Z) \times \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}}(X, Z)$$

Now, what we have seen above is that $\underline{\text{Cat}}$ can be seen as a 2-category. For any two small categories $\mathcal{C}$ and $\mathcal{D}$, we have a category $\text{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are functors $F: \mathcal{C} \rightarrow \mathcal{D}$, and whose morphisms are natural transformations. In that context, composition is the vertical composition of natural transformation. On the other hand, given a third small category $\mathcal{E}$, the composition

$$\text{Hom}_{\underline{\text{Cat}}}(\mathcal{D}, \mathcal{E}) \times \text{Hom}_{\underline{\text{Cat}}}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Hom}_{\underline{\text{Cat}}}(\mathcal{C}, \mathcal{E})$$

sends a pair of functors $(H: \mathcal{D} \rightarrow \mathcal{E}, F: \mathcal{C} \rightarrow \mathcal{D})$ to the composition $H \circ F$ and can be extended to natural transformations by sending a pair of natural transformation $(\beta: H \Rightarrow K, \alpha: F \Rightarrow G)$ to the horizontal composition $\beta \circ_h \alpha$.

We will not explore this notion any further in those notes, but the interested reader can read [Mac71, Chapter XII.3].

2.3 Equivalence of categories

We have seen in Definition 2.1.5 how two small categories can be isomorphic. However, isomorphisms of categories is too restrictive of a notion. Intuitively, we want to say that two categories $\mathcal{C}$ and $\mathcal{D}$ are "the same" if we can find two functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ that are inverses to each other. However, we know that it is meaningless in general to check equality for functors. In particular, we should not ask that $F \circ G$ be equal to the identity of $\mathcal{D}$. Instead, what we can meaningfully impose is that the two functors $F \circ G$ and $\text{Id}_{\mathcal{D}}$ be *isomorphic*. This leads to the richer notion of **equivalence of categories**.

Definition 2.3.1. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is an **equivalence of categories** if there exists some functor $G: \mathcal{D} \rightarrow \mathcal{C}$, and two natural isomorphisms $\alpha: F \circ G \simeq \text{Id}_{\mathcal{D}}$ and $\beta: G \circ F \simeq \text{Id}_{\mathcal{C}}$.

Example 2.3.2. 1. Consider the two small categories, $\mathcal{C}$ and $\mathcal{D}$ given by the two following graphs.

$$A \curvearrowright \text{Id}_A \tag{2.4}$$

$$\text{Id}_B \hookrightarrow B \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{f^{-1}} \end{array} C \curvearrowright \text{Id}_C \tag{2.5}$$

We will prove that they are equivalent. Consider $F: \mathcal{C} \rightarrow \mathcal{D}$ sending A to B , and Id_A to Id_B , and $G: \mathcal{D} \rightarrow \mathcal{C}$ sending all objects to A and all morphisms to Id_A . The composition $G \circ F$ is the identity of $\mathcal{C}$, so we can pick $\beta = \text{Id}_{\text{Id}_{\mathcal{C}}}$. On the other hand, the composition $F \circ G$ is the functor $\mathcal{D} \rightarrow \mathcal{D}$ sending both objects to B and sending all maps to Id_B . Let us define α as $\alpha_B = \text{Id}_B: F \circ G(B) = B \rightarrow B$, and $\alpha_C = f: F \circ G(C) = B \rightarrow C$. By construction, f is an isomorphism, hence to deduce that α is a natural isomorphism, we just need to prove that it is a natural transformation. We will check the naturality condition for

both non-identity maps, since it holds automatically for identity maps. Consider the square

$$\begin{array}{ccc} F \circ G(B) & \xrightarrow{\alpha_B} & B \\ F \circ G(f) \downarrow & & \downarrow f \\ F \circ G(C) & \xrightarrow{\alpha_C} & C \end{array}$$

Unfolding the definitions, we see that the top and left maps are both Id_B , while the bottom and right maps are both f . Hence, the square is indeed commutative. For the map $f^{-1}: C \rightarrow B$, we have

$$\begin{array}{ccc} F \circ G(C) & \xrightarrow{\alpha_C} & C \\ F \circ G(f^{-1}) \downarrow & & \downarrow f^{-1} \\ F \circ G(B) & \xrightarrow{\alpha_B} & B \end{array}$$

Now, we see that the top and right maps are f and f^{-1} respectively, hence they compose to Id_B . On the other hand, the left and bottom maps are both Id_B . Hence, the square is commutative. We have proven that $\alpha: F \circ G \rightarrow \text{Id}_{\mathcal{D}}$ is a natural transformation, taking values in isomorphisms, hence it is a natural isomorphism.

- Consider the functor $F: \underline{\mathbf{k}\text{-Mat}} \rightarrow \underline{\mathbf{k}\text{-Vect}_{\text{fin}}}$ from Exercise 18. We will show that it is an equivalence of categories. To define a functor $G: \underline{\mathbf{k}\text{-Vect}_{\text{fin}}} \rightarrow \underline{\mathbf{k}\text{-Mat}}$, first, for each finite dimensional vector space V , pick a basis of vectors $(e_1, \dots, e_{\dim(V)})$. Make sure to pick the canonical basis for k^n , for each $n \geq 0$. Then define G as the functor sending V to $\dim(V)$, and sending a linear map $\varphi: V \rightarrow W$ to the matrix of φ in the chosen basis for V and W , $(e_1, \dots, e_{\dim(V)})$, and $(f_1, \dots, f_{\dim(W)})$. Now, observe that $G \circ F = \text{Id}_{\underline{\mathbf{k}\text{-Mat}}}$, thus, we only have to find a natural isomorphism $\beta: F \circ G \rightarrow \text{Id}$. Let V be a vector space, with its chosen basis $(e_1, \dots, e_{\dim(V)})$. Define $\beta_V: k^{\dim(V)} \rightarrow V$ as the linear map sending the i -th vector in the canonical basis of $k^{\dim(V)}$ to e_i . Let us show that β is a natural transformation. Let $\varphi: V \rightarrow W$ be a linear map. Consider the diagram

$$\begin{array}{ccc} F \circ G(V) & \xrightarrow{\beta_V} & V \\ F \circ G(\varphi) \downarrow & & \downarrow \varphi \\ F \circ G(W) & \xrightarrow{\beta_W} & W \end{array}$$

Let us show that it is commutative. Let $v_i \in k^{\dim(V)}$ be the i -th vector in the canonical basis of $k^{\dim(V)}$. The top right composition sends it to $\varphi(\beta_V(v_i)) = \varphi(e_i)$. On the other hand $F \circ G(\varphi)(v_i) = \sum_{j=1}^{\dim(V)} m_{i,j} w_j$, where the $m_{i,j}$ are the coefficients of the matrix of φ in the chosen basis for V and W , and w_j is the j -th vector in the canonical basis of $k^{\dim(W)}$. Now, observe that $\beta_W(F \circ G(\varphi)(v_i)) = \sum_{j=1}^{\dim(W)} m_{i,j} f_j$. But, since the $m_{i,j}$ are

the coefficients of the matrix of φ , we must have $\beta_W(F \circ G(\varphi)(v_i)) = \varphi(e_i)$, hence the diagram is commutative. Finally, β is a natural transformation, and since β takes values in isomorphisms, by definition, it must be a natural isomorphism. Hence F and G are equivalences of categories.

Remark 2.3.3. Note that $\mathbf{k}\text{-Mat}$ is a small category, while $\mathbf{k}\text{-Vect}_{\text{fin}}$ is not. The fact that $\mathbf{k}\text{-Vect}_{\text{fin}}$ is equivalent to a small category is a property of $\mathbf{k}\text{-Vect}_{\text{fin}}$ that is not always true of an arbitrary category. In that case, we say that $\mathbf{k}\text{-Vect}_{\text{fin}}$ is **essentially small**.

Definition 2.3.4. Let $\mathcal{C}$ be a category. We say that $\mathcal{C}$ is **essentially small** if there exists a small category $\mathcal{D}$ and an equivalence of categories $F: \mathcal{C} \rightarrow \mathcal{D}$.

Exercise 24. Let G and H be two groups. What is an equivalence of categories $F: \underline{G} \rightarrow \underline{H}$. Same question for pre-ordered sets seen as categories.

Exercise 25. Let $\mathcal{C}$ and $\mathcal{C}'$ be two small categories and let $\mathcal{D}$ and $\mathcal{D}'$ be two categories. Assume that $\mathcal{C}$ and $\mathcal{C}'$ are equivalent and that $\mathcal{D}$ and $\mathcal{D}'$ are equivalent. Show that there exists an equivalence

$$\text{Fun}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}(\mathcal{C}', \mathcal{D}')$$

2.3.1 Characterizing equivalences of categories

We have defined an equivalence of categories as a functor admitting an inverse. Conceptually, this is satisfying, but it can be practically inconvenient. Indeed, it will often be the case that we only have a use for one direction of an equivalence of categories. Hence, it is convenient to have a way of checking that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is part of an equivalence, without necessarily exhibiting an inverse. This is the content of the following proposition. Note that its proof relies on a (strong) version of the axiom of choice, and thus its use may not be satisfactory to people preferring to work with more constructive approach as much as possible. The proposition itself will not be used in the rest of those notes, but it is very convenient to keep it in mind when working through examples. It may serve as a first step to check whether an inverse exist before trying to exhibit it explicitly.

Proposition 2.3.5. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Assume that

- F is fully faithful,
- F is essentially surjective.

Then, F is an equivalence of categories.

Proof. The proof will rely on a very strong version of the axiom of choice. Let us immediately state its use. Since, F is essentially surjective, for all objects X in $\mathcal{D}$, there exists some object C_X in $\mathcal{C}$ and an isomorphism $\varphi_X: F(C_X) \rightarrow X$ in $\mathcal{D}$. Assume that we can **choose simultaneously** some C_X and φ_X for all

objects X of $\mathcal{D}$. This amounts to asking for the existence of a choice function from a class to a product of classes.

For the remainder of this proof, assume that such a choice of C_X and φ_X has been made for all objects X of $\mathcal{D}$. Let us now define a functor $G: \mathcal{D} \rightarrow \mathcal{C}$. On objects, it will send X to C_X . Let $f: X \rightarrow Y$ be a map in $\mathcal{D}$, and consider the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi_X \uparrow & & \downarrow \varphi_Y^{-1} \\ F \circ G(X) & \xrightarrow{\varphi_Y^{-1} \circ f \circ \varphi_X} & F \circ G(Y) \end{array}$$

By assumption, the functor F is fully faithful, hence the map

$$F: \text{Hom}_{\mathcal{C}}(G(X), G(Y)) \rightarrow \text{Hom}_{\mathcal{D}}(F \circ G(X), F \circ G(Y))$$

is a bijection. Thus, we can unambiguously define $G(f)$ to be the preimage of $\varphi_Y^{-1} \circ f \circ \varphi_X$ under this map. (It may be helpful to think of $G(f)$ as $F^{-1}(\varphi_Y^{-1} \circ f \circ \varphi_X)$.)

Let us now prove that G is indeed a functor. First, if X is an object of $\mathcal{D}$, then $\varphi_X^{-1} \circ \text{Id}_X \circ \varphi_X$ is simply $\text{Id}_{F(C_X)}$. Since F is a functor, it sends identity to identity, thus $G(\text{Id}_X) = \text{Id}_{C_X}$. For composition, let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be two composable maps in $\mathcal{D}$. Then, consider the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{\text{Id}_Y} & Y & \xrightarrow{g} & Z \\ \varphi_X \uparrow & & \searrow \varphi_Y^{-1} & & \nearrow \varphi_Y & & \downarrow \varphi_Z^{-1} \\ F \circ G(X) & \xrightarrow{\varphi_Y^{-1} \circ f \circ \varphi_X} & F \circ G(Y) & \xrightarrow{\varphi_Z^{-1} \circ g \circ \varphi_Y} & F \circ G(Z) & & \end{array}$$

The bottom arrows of this diagram correspond to $F \circ G(f)$ and $F \circ G(g)$ respectively, by definition of G . On the other hand, commutativity of the diagram implies that their composition gives $F \circ G(g \circ f)$. Thus, we have

$$\begin{aligned} F(G(g \circ f)) &= F \circ G(g \circ f) \\ &= F \circ G(g) \circ F \circ G(f) \\ &= F(G(g)) \circ F(G(f)) \\ &= F(G(g) \circ G(f)) \end{aligned}$$

But now, faithfulness of F implies that $G(g \circ f) = G(g) \circ G(f)$, and we have proven that G is a functor.

Finally, we need to find two natural isomorphism $\alpha: F \circ G \rightarrow \text{Id}_{\mathcal{D}}$, and $\beta: G \circ F \rightarrow \text{Id}_{\mathcal{C}}$. For α , set $\alpha_X = \varphi_X: F(G(X)) = F(C_X) \rightarrow X$. This takes values into isomorphisms, by definition. For naturality, let $f: X \rightarrow Y$ be a map

in $\mathcal{D}$, consider the diagram

$$\begin{array}{ccc} F \circ G(X) & \xrightarrow{\varphi_X} & X \\ F \circ G(f) \downarrow & & \downarrow f \\ F \circ G(Y) & \xrightarrow{\varphi_Y} & Y \end{array}$$

and observe that G was defined specifically to make this diagram commutative. Hence, α is a natural isomorphism.

For the other direction, let C be some object in $\mathcal{C}$, and consider

$$F: \text{Hom}_{\mathcal{C}}(G \circ F(C), C) \rightarrow \text{Hom}_{\mathcal{D}}(F \circ G \circ F(C), F(C))$$

Since F is fully-faithful, this map is a bijection, hence, there is a unique preimage to the isomorphism $\varphi_{F(C)}: F \circ G \circ F(C) \rightarrow F(C)$. Let $\beta_C: G \circ F(C) \rightarrow C$ be this preimage. Let us show that β is a natural transformation. Let $g: C \rightarrow D$ be a map in $\mathcal{C}$, and consider the diagram

$$\begin{array}{ccc} G \circ F(C) & \xrightarrow{\beta_C} & C \\ G \circ F(g) \downarrow & & \downarrow g \\ G \circ F(D) & \xrightarrow{\beta_D} & D \end{array}$$

We need to show that $g \circ \beta_C = \beta_D \circ (G \circ F(g))$. First, apply F to the entire diagram. We get

$$\begin{array}{ccc} F \circ G \circ F(C) & \xrightarrow{F(\beta_C)} & F(C) \\ F \circ G(F(g)) \downarrow & & \downarrow F(g) \\ F \circ G \circ F(D) & \xrightarrow{F(\beta_D)} & F(D) \end{array}$$

Now, recall that $F(\beta_C) = \varphi_{F(C)} = \alpha_{F(C)}$, by definition of β , and the same holds for D , hence the above diagram commute by naturality of α . Hence, we have $F(g \circ \beta_C) = F(\beta_D \circ (G \circ F(g)))$. But since F is faithful, we deduce the desired equality, hence β is a natural transformation.

The only thing that remains to be proven is that β takes values in isomorphism. Let C be an object in $\mathcal{C}$. Consider the map

$$F: \text{Hom}_{\mathcal{C}}(C, G \circ F(C)) \rightarrow \text{Hom}_{\mathcal{D}}(F(C), F \circ G \circ F(C))$$

Since F is fully-faithful, we can define $\gamma_C: C \rightarrow G \circ F(C)$ as the only preimage of φ_C^{-1} under F . And now, observe that $F(\beta_C \circ \gamma_C) = \text{Id}_{F(C)}$ and that $F(\gamma_C \circ \beta_C) = \text{Id}_{F \circ G \circ F(C)}$. Since F is faithful, we deduce that $\beta_C \circ \gamma_C = \text{Id}_C$ and $\gamma_C \circ \beta_C = \text{Id}_{G \circ F(C)}$. In particular, β_C is an isomorphism. $\square$

Exercise 26. Prove the converse statement of Proposition 2.3.5.

Chapter 3

Adjunctions

3.1 Definitions

Definition 3.1.1. An *adjunction* is the data of

- a pair of **adjoint functors** $F: \mathcal{C} \rightarrow \mathcal{D}$, $G: \mathcal{D} \rightarrow \mathcal{C}$.
- for all objects X in $\mathcal{C}$, Y in $\mathcal{D}$, a bijection

$$\theta_{X,Y}: \text{Hom}_{\mathcal{C}}(X, G(Y)) \simeq \text{Hom}_{\mathcal{D}}(F(X), Y)$$

Such that θ is **natural in X and Y** . More precisely, for all object X in $\mathcal{C}$, $\theta_{X,-}$ is a natural transformation between the functors $\text{Hom}_{\mathcal{C}}(X, G(-)): \mathcal{D} \rightarrow \underline{\text{Set}}$ and $\text{Hom}_{\mathcal{D}}(F(X), -): \mathcal{D} \rightarrow \underline{\text{Set}}$, and for all objects Y in $\mathcal{D}$, $\theta_{-,Y}$ is a natural transformation between the functors $\text{Hom}_{\mathcal{C}}(-, G(Y)): \mathcal{C}^{\text{op}} \rightarrow \underline{\text{Set}}$ and $\text{Hom}_{\mathcal{D}}(F(-), Y): \mathcal{C}^{\text{op}} \rightarrow \underline{\text{Set}}$.

In that case, we will say that F and G are an **adjoint pair**, and write $F \dashv G$. We will say that F is the **left adjoint** to G and that G is the **right adjoint** to F .

Remark 3.1.2. The above definition can be made more explicit with the use of product categories. Given two categories, in this case $\mathcal{C}^{\text{op}}$ and $\mathcal{D}$, the product of the two categories, $\mathcal{C}^{\text{op}} \times \mathcal{D}$ has, for objects, pairs of objects (X, Y) where X is an object of $\mathcal{C}^{\text{op}}$ and Y is an object of $\mathcal{D}$. And, given (X, Y) and (X', Y') two objects in $\mathcal{C}^{\text{op}} \times \mathcal{D}$, a morphism from (X, Y) to (X', Y') is the data of a morphism $f^{\text{op}}: X \rightarrow X'$ in $\mathcal{C}^{\text{op}}$ and a morphism $g: Y \rightarrow Y'$ in $\mathcal{D}$. Identity and compositions are computed component wise. Now, we can rephrase the condition for an adjunction by asking that θ is a natural transformation between two functors :

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(-, G(-)): \mathcal{C}^{\text{op}} \times \mathcal{D} &\rightarrow \underline{\text{Set}} \\ (X, Y) &\mapsto \text{Hom}_{\mathcal{C}}(X, G(Y)) \\ (f^{\text{op}}, g) &\mapsto \text{Hom}_{\mathcal{C}}(f, G(Y')) \circ \text{Hom}_{\mathcal{C}}(X, G(g)) \end{aligned}$$

and

$$\begin{aligned} \text{Hom}_{\mathcal{D}}(F(-), -) \mathcal{C}^{\text{op}} \times \mathcal{D} &\rightarrow \underline{\text{Set}} \\ (X, Y) &\mapsto \text{Hom}_{\mathcal{D}}(F(X), G(Y)) \\ (f^{\text{op}}, g) &\mapsto \text{Hom}_{\mathcal{D}}(F(f), Y') \circ \text{Hom}_{\mathcal{D}}(F(X), g) \end{aligned}$$

Alternative Definition 3.1.3. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ be two functors. They form an **adjoint pair**, $F \dashv G$ if there exists two natural transformations,

- $\epsilon: F \circ G \rightarrow \text{Id}_{\mathcal{D}}$, called the **co-unit**,
- $\eta: \text{Id}_{\mathcal{C}} \rightarrow G \circ F$ called the **unit**.

such that the following two triangles commute

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & F \circ G \circ F \\ & \searrow \text{Id}_F & \downarrow \epsilon F \\ & & F \end{array} \quad (3.1)$$

$$\begin{array}{ccc} G & \xrightarrow{\eta G} & G \circ F \circ G \\ & \searrow \text{Id}_G & \downarrow G\epsilon \\ & & G \end{array} \quad (3.2)$$

Proposition 3.1.4. Definitions 3.1.1 and 3.1.3 are equivalent.

Proof. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ be given.

- ($\Rightarrow$) Assume that a natural isomorphism θ is given. For X an object of $\mathcal{C}$, define $\eta_X: X \rightarrow G \circ F(X)$ as $\eta_X = \theta_{X, F(X)}^{-1}(\text{Id}_{F(X)})$. Let us prove that η is a natural transformation. Let $f: X \rightarrow Y$ be a map in $\mathcal{C}$. We need to prove that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & G \circ F(X) \\ f \downarrow & & \downarrow G \circ F(f) \\ Y & \xrightarrow{\eta_Y} & G \circ F(Y) \end{array}$$

Specifically, we need to prove

$$\eta_Y \circ f = G \circ F(f) \circ \eta_X \in \text{Hom}_{\mathcal{C}}(X, G \circ F(Y)). \quad (3.3)$$

To do so, we will use the naturality of θ in both variables. First, consider the naturality of $\theta_{X, -}$, with respect to the map $F(f): F(X) \rightarrow F(Y)$. This gives the commutativity of the following diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X, G \circ F(X)) & \xrightarrow{\theta_{X, F(X)}} & \text{Hom}_{\mathcal{D}}(F(X), F(X)) \\ \text{Hom}_{\mathcal{C}}(X, G \circ F(f)) \downarrow & & \downarrow \text{Hom}_{\mathcal{D}}(F(Y), F(f)) \\ \text{Hom}_{\mathcal{C}}(X, G \circ F(Y)) & \xrightarrow{\theta_{X, F(Y)}} & \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \end{array}$$

Starting with η_X at the top left gives $\text{Id}_{F(X)}$ on the top right, and hence $F(f)$ in the bottom right. On the other hand, we get $G \circ F(f) \circ \eta_X$ on the bottom left. Thus, we deduce

$$\theta_{X,F(Y)}(G \circ F(f) \circ \eta_X) = F(f)$$

On the other hand, we consider the naturality of $\theta_{-,F(Y)}$ with respect to the map $f: X \rightarrow Y$. This gives the commutative diagram:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(Y, G \circ F(Y)) & \xrightarrow{\theta_{Y,F(Y)}} & \text{Hom}_{\mathcal{D}}(F(Y), F(Y)) \\ \text{Hom}_{\mathcal{C}}(f, G \circ F(Y)) \downarrow & & \downarrow \text{Hom}_{\mathcal{D}}(F(f), F(Y)) \\ \text{Hom}_{\mathcal{C}}(X, G \circ F(Y)) & \xrightarrow{\theta_{X,F(Y)}} & \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \end{array}$$

starting with η_Y in the top left gives $\text{Id}_{F(Y)}$ in the top right corner, and $F(f)$ in the bottom right. On the other, we get $\eta_Y \circ f$ in the bottom left, hence we have the equality

$$\theta_{X,F(Y)}(\eta_Y \circ f) = F(f)$$

putting both previous equations together, we get

$$\theta_{X,F(Y)}(G \circ F(f) \circ \eta_X) = \theta_{X,F(Y)}(\eta_Y \circ f)$$

But, since $\theta_{X,F(Y)}$ is a bijection, we deduce equality (3.3). Hence, η is a natural transformation.

Similarly, for Y an object of $\mathcal{D}$, define $\epsilon_Y: F \circ G(Y) \rightarrow Y$ as $\epsilon_Y = \theta_{G(Y),Y}(\text{Id}_{G(Y)})$.

Exercise 27. Prove that ϵ is a natural transformation.

It remains to be proven that for all X objects of $\mathcal{C}$, and for all Y objects of $\mathcal{D}$, the compositions $\epsilon_{F(X)} \circ F(\eta_X): F(X) \rightarrow F(X)$ and $G(\epsilon_Y) \circ \eta_{G(Y)}: G(Y) \rightarrow G(Y)$ are equal to $\text{Id}_{F(X)}$ and $\text{Id}_{G(Y)}$ respectively. For the first equality, consider the commutative square coming from the naturality of $\theta_{-,F(X)}$ with respect to the map $\eta_X: X \rightarrow G \circ F(X)$.

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(G \circ F(X), G \circ F(X)) & \xrightarrow{\theta_{G \circ F(X),F(X)}} & \text{Hom}_{\mathcal{D}}(F \circ G \circ F(X), F(X)) \\ \text{Hom}_{\mathcal{C}}(\eta_X, G \circ F(X)) \downarrow & & \downarrow \text{Hom}_{\mathcal{D}}(F(\eta_X), F(X)) \\ \text{Hom}_{\mathcal{C}}(X, G \circ F(X)) & \xrightarrow{\theta_{X,F(X)}} & \text{Hom}_{\mathcal{D}}(F(X), F(X)) \end{array}$$

Starting with $\text{Id}_{G \circ F(X)}$ in the top left gives $\epsilon_{F(X)}$ in the top right, and $F(\eta_X) \circ \epsilon_{F(X)}$ in the bottom right. On the other hand, we get η_X in the bottom left. Since, by construction, $\theta_{X,F(X)}(\eta_X) = \text{Id}_{F(X)}$, we deduce the equality

$$F(\eta_X) \circ \epsilon_{F(X)} = \text{Id}_{F(X)}$$

We leave the rest of the proof as an exercise.

Exercise 28. Let Y be an object in $\mathcal{D}$. Prove that

$$G(\epsilon_Y) \circ \eta_{G(Y)} = \text{Id}_{G(Y)}$$

- ($\Leftarrow$) Assume that the natural transformation $\eta: \text{Id}_{\mathcal{C}} \rightarrow G \circ F$ and $\epsilon: F \circ G \rightarrow \text{Id}_{\mathcal{D}}$ are given, with the appropriate relations. Let X and Y be two objects in $\mathcal{C}$ and $\mathcal{D}$ respectively. Define

$$\begin{aligned} \theta_{X,Y}: \text{Hom}_{\mathcal{C}}(X, G(Y)) &\rightarrow \text{Hom}_{\mathcal{D}}(F(X), Y) \\ f: X \rightarrow G(Y) &\mapsto \epsilon_Y \circ F(f): F(X) \rightarrow F \circ G(Y) \rightarrow Y \end{aligned}$$

$$\begin{aligned} \pi_{X,Y}: \text{Hom}_{\mathcal{D}}(F(X), Y) &\rightarrow \text{Hom}_{\mathcal{C}}(X, G(Y)) \\ g: F(X) \rightarrow Y &\mapsto G(g) \circ \eta_X: X \rightarrow G \circ F(X) \rightarrow G(Y) \end{aligned}$$

Let us first show that $\theta_{X,Y}$ and $\pi_{X,Y}$ are inverse bijection. Let $f: X \rightarrow G(Y)$ be a map in $\mathcal{C}$, and consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & G \circ F(X) \\ f \downarrow & & \downarrow G \circ F(f) \\ G(Y) & \xrightarrow{\eta_{G(Y)}} & G \circ F \circ G(Y) \\ & \searrow \text{Id}_{G(Y)} & \downarrow G(\epsilon_Y) \\ & & G(Y) \end{array}$$

By definition, $\pi_{X,Y}(\theta_{X,Y}(f)) = \pi_{X,Y}(\epsilon_Y \circ F(f)) = G(\epsilon_Y) \circ G(F(f)) \circ \eta_X$, which is the top right composition in the above diagram. But note that the top square in the diagram is commutative, by naturality of η , and the bottom triangle is commutative, by hypothesis on F and G . Hence, the entire diagram must be commutative, and we have $\pi_{X,Y}(\theta_{X,Y}(f)) = f$.

Exercise 29. Show that for any map $g: F(X) \rightarrow Y$, $\theta_{X,Y}(\pi_{X,Y}(g)) = g$.

With the exercise, we get that $\theta_{X,Y}$ is an isomorphism for all X and Y . Thus, the only thing that remains to be proven is that θ is a natural transformation. Let $h^{\text{op}}: X \rightarrow Z$ be a map in $\mathcal{C}^{\text{op}}$, and $k: Y \rightarrow W$ be a map in $\mathcal{D}$. We need to prove that the following diagram is commutative.

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X, G(Y)) & \xrightarrow{\theta_{X,Y}} & \text{Hom}_{\mathcal{D}}(F(X), Y) \\ \text{Hom}_{\mathcal{C}}(h, G(k)) \downarrow & & \downarrow \text{Hom}_{\mathcal{D}}(F(h), k) \\ \text{Hom}_{\mathcal{C}}(Z, G(W)) & \xrightarrow{\theta_{Z,W}} & \text{Hom}_{\mathcal{D}}(F(Z), W) \end{array}$$

starting with $f: X \rightarrow G(Y)$ in the top left, we get $\epsilon_Y \circ F(f)$ in the top right corner, and $k \circ \epsilon_Y \circ F(f) \circ F(h)$ in the bottom right corner. On the other hand, we get $G(k) \circ f \circ h$ in the bottom left corner, which gives

$\epsilon_W \circ F(G(k)) \circ F(f) \circ F(h)$ in the bottom right corner. But now, naturality of ϵ with respect to k gives

$$\epsilon_W \circ F(G(k)) \circ F(f) \circ F(h) = k \circ \epsilon_Y \circ F(f) \circ F(h)$$

This proves that the square is commutative, hence that θ is a natural isomorphism, which concludes the proof. $\square$

3.2 Examples

We have already seen many examples of forgetful functors in Example 2.1.2. Most of them are part of a forgetful functor/free functor adjunction. Let us detail a few of those.

Example 3.2.1. Consider the functor $U: \underline{\mathbf{Grp}} \rightarrow \underline{\mathbf{Set}}$, sending a group to its underlying set. It admits a left adjoint, $F: \underline{\mathbf{Set}} \rightarrow \underline{\mathbf{Grp}}$, defined as follows. For any set X , $F(X)$ is the **free group** with generators X . That is, an element of $F(X)$ is a word with letters either elements of X or of the form x^{-1} , with $x \in X$. More explicitly, elements of $F(X)$ are families $(a_1, \dots, a_n)$, with $a_i \in X$ or $a_i = x^{-1}$, with $x \in X$, for $1 \leq i \leq n$, together with the relation $xx^{-1} = x^{-1}x$ is equal to the empty word. The product in $F(X)$ is given by concatenation of the words, and the neutral element of $F(X)$ is the empty word.

Exercise 30. Check that $F(X)$ is indeed a group.

Given $f: X \rightarrow Y$ a map in $\underline{\mathbf{Set}}$, $F(f): F(X) \rightarrow F(Y)$ sends a word $(x_1, \dots, x_n)$ to the word $(f(x_1), \dots, f(x_n))$ in $F(Y)$, with the convention that $f(x^{-1}) = f(x)^{-1}$ for all $x \in X$. Now, let us exhibit the natural transformation θ . Let X be a set, let G be a group, and let $f: X \rightarrow U(G)$ be a set-theoretic map. Define $\theta_{X,G}(f): F(X) \rightarrow G$ via the formula

$$\theta_{X,G}(f)(x_1, \dots, x_n) = f(x_1) \star \dots \star f(x_n),$$

where $\star$ stands for the group operation in G , and $f(x^{-1}) = f(x)^{-1}$, for all $x \in X$. By construction, $\theta_{X,G}(f)$ is indeed a group homomorphism. Let us define its inverse as follows. Let $g: F(f) \rightarrow G$ be a map in $\underline{\mathbf{Grp}}$. Define

$$\theta_{X,G}^{-1}(g)(x) = f(x)$$

Then, we can check that $\theta_{X,G}^{-1}$ is indeed an inverse to $\theta_{X,G}$. Let $f: X \rightarrow U(G)$ be a map, and $x \in X$. We compute

$$\begin{aligned} \left(\theta_{X,G}^{-1}(\theta_{X,G}(f)) \right) (x) &= (\theta_{X,G}(f))(x) \\ &= f(x) \end{aligned}$$

Hence, $\theta_{X,G}^{-1} \circ \theta_{X,G} = \text{Id}_{\text{Hom}_{\text{Set}}(X, U(G))}$. On the other hand, given $g: F(X) \rightarrow G$, and $(x_1, \dots, x_n)$ a word in $F(X)$, we have

$$\begin{aligned} \left(\theta_{X,G}(\theta_{X,G}^{-1}(g)) \right) (x_1, \dots, x_n) &= \theta_{X,G}^{-1}(g)(x_1) \star \dots \star \theta_{X,G}^{-1}(g)(x_n) \\ &= g(x_1) \star \dots \star g(x_n) \end{aligned}$$

But since g is assumed to be a group homomorphism, and since the group operation in $F(X)$ is concatenation, this last line must be equal to $g(x_1, \dots, x_n)$. Hence, we have $\theta_{X,G}(\theta_{X,G}^{-1}(g)) = g$. And $\theta_{X,G}$ and $\theta_{X,G}^{-1}$ are indeed inverses to each other.

Exercise 31. Show that $\theta_{-, -}$ is a natural transformation.

Example 3.2.2. Now, consider the functor $U: \underline{\text{Ab}} \rightarrow \underline{\text{Grp}}$, sending an abelian group to itself, in the category of all groups. This can also be interpreted as a forgetful functor, and as such, should admit a left adjoint. Indeed its left adjoint is the abelianization functor $(-)^{\text{ab}}: \underline{\text{Grp}} \rightarrow \underline{\text{Ab}}$, sending a group G to its quotient $G^{\text{ab}} = G/\sim$, where $\sim$ is the equivalence relation generated by $gh \sim hg$ for all $g, h \in G$. Now, consider the following well-known algebraic statement (which is not too hard to prove by hand).

Lemma 3.2.3. Let $f: G \rightarrow A$ be a group homomorphism from a group G to an abelian group A . Then, f factors uniquely through some map f^{ab} as follows

$$\begin{array}{ccc} G & \xrightarrow{f} & A \\ & \searrow \pi & \nearrow f^{\text{ab}} \\ & G^{\text{ab}} & \end{array}$$

where π is the projection to the quotient.

We can easily deduce from it that $(-)^{\text{ab}} \dashv U$ form an adjoint pair. Indeed, it suffices to define $\theta_{G,A}(f) = f^{\text{ab}}$, and $\theta_{G,A}^{-1}(g) = g \circ \pi$.

Exercise 32. Check that the pair $(-)^{\text{ab}} \dashv U$ is indeed an adjunction.

Example 3.2.4. Let us spend some time on a fundamental example. As we have seen in Alternative Definition 1.2.3, small categories can be thought of as graphs with additional structure. Hence, there should exist some forgetful functor from the category of small categories to that of graphs, $U: \underline{\text{Cat}} \rightarrow \underline{\text{Graph}}$. Let us first define a suitable category of graphs, then we will show that we have an adjunction between the two categories. Let $\underline{\text{Graph}}$ be the category such that

- its objects are directed graphs, that is, an object G is the data of
 - two sets $V(G)$ and $A(G)$ of vertices and arrows respectively,
 - two maps $s_G, t_G: A(G) \rightarrow V(G)$, mapping each arrow to its source vertex and target vertex respectively.
- Given two graphs G and H , a morphism $f \in \text{Hom}_{\underline{\text{Graph}}}(G, H)$ is the data of

- a map of sets $f_V: V(G) \rightarrow V(H)$
- a map of sets $f_A: A(G) \rightarrow A(H)$

such that $s_H \circ f_A = f_V \circ s_G$ and $t_H \circ f_A = f_V \circ t_G$.

- given a graph G , $\text{Id}_G: G \rightarrow G$ is the identity on both arrows and vertices,
- composition of maps $f: G \rightarrow H$ and $g: H \rightarrow K$ is defined via $(g \circ f)_A = g_A \circ f_A$ and $(g \circ f)_V = g_V \circ f_V$.

Note that there are many other categories of graphs, but this is the main one we will be interested in during this class.

Now, we can define the forgetful functor $U: \underline{\text{Cat}} \rightarrow \underline{\text{Graph}}$. Based on Alternative Definition 1.2.3, given any small category $\mathcal{C}$, we can define $U(\mathcal{C})$ as the underlying small category, that is $V(U(\mathcal{C})) = \text{obj}(\mathcal{C})$, and $A(U(\mathcal{C}))$ is the set of morphisms of $\mathcal{C}$. $s_{U(\mathcal{C})}$ and $t_{U(\mathcal{C})}$ are given by domains and codomains of maps respectively. On functors, $G: \mathcal{C} \rightarrow \mathcal{D}$, define $U(G): U(\mathcal{C}) \rightarrow U(\mathcal{D})$ via $U(G)_V = G_{\text{obj}}: \text{obj}(\mathcal{C}) \rightarrow \text{obj}(\mathcal{D})$, and $U(G)_A$ is the action on morphisms of the functor G .

Let us now prove that U admits a left adjoint, $F: \underline{\text{Graph}} \rightarrow \underline{\text{Cat}}$. Let G be a graph, and define $F(G)$ as the small category with

- Objects of $F(G)$ are vertices of G .
- morphisms of $F(G)$ are given by the data of
 - some $n \geq 0$,
 - a family of objects $(X_0, \dots, X_n)$,
 - for all $1 \leq i \leq n$, an arrow $f_i \in A(G)$ such that $s_G(f_i) = X_{i-1}$, and $t_G(f_i) = X_i$.

We will write such a morphism $(X_0, f_1, X_1, \dots, f_n, X_n)$, and we have $(X_0, f_1, X_1, \dots, f_n, X_n) \in \text{Hom}_{F(G)}(X_0, X_n)$.

- Given X an object of $F(G)$, the identity of X , Id_X is the unique morphism with $n = 0$, $(X) \in \text{Hom}_{F(G)}(X, X)$.
- Given two morphisms in $F(G)$, $(X_0, f_1, X_1, \dots, f_n, X_n) \in \text{Hom}_{F(G)}(X_0, X_n)$ and $(Y_0, g_1, Y_1, \dots, g_m, Y_m) \in \text{Hom}_{F(G)}(Y_0, Y_m)$, they are composable if and only if $X_n = Y_0$. In that case, define their composition as the concatenation

$$\begin{aligned} (Y_0, g_1, Y_1, \dots, g_m, Y_m) \circ (X_0, f_1, X_1, \dots, f_n, X_n) \\ = (X_0, f_1, X_1, \dots, f_n, X_n, g_1, Y_1, \dots, g_m, Y_m). \end{aligned}$$

Exercise 33. Check that $F(G)$ is indeed a small category.

Now, assume that $g: G \rightarrow H$ is a map in $\mathbf{Graph}$. And define the morphism in $\mathbf{Cat}$, $F(g): F(G) \rightarrow F(H)$ as follows. On objects, $F(g)(X) = g_V(X)$, for each $X \in \text{obj}(F(G)) = V(G)$. Given a morphism $(X_0, f_1, X_1, \dots, f_n, X_n) \in \text{Hom}_{F(G)}(X_0, X_n)$, define

$$F(g)(X_0, f_1, X_1, \dots, f_n, X_n) = (g_V(X_0), g_A(f_1), g_V(X_1), \dots, g_A(f_n), g_V(X_n))$$

The condition $s_H \circ g_A = g_V \circ s_G$, and the same for targets, guarantees that $F(g)$ sends morphisms in $F(G)$ to morphisms in $F(H)$. Since $F(g)$ furthermore preserves concatenations, $F(g): F(G) \rightarrow F(H)$ is indeed a functor, hence a morphism in $\mathbf{Cat}$.

Exercise 34. Check that $F: \mathbf{Graph} \rightarrow \mathbf{Cat}$ is indeed a functor.

Let us now prove that we have an adjunction $F \dashv U$. For a graph G , we start by defining the unit $\eta_G: G \rightarrow U(F(G))$. On vertices, $(\eta_G)_V(X) = X$. On arrows, $(\eta_G)_A(f) = (s_G(f), f, t_G(f))$. Now, for a small category $\mathcal{C}$, define the co-unit as follows

$$\begin{aligned} \epsilon_{\mathcal{C}}: F(U(\mathcal{C})) &\rightarrow \mathcal{C} \\ X &\mapsto X \\ (X_0, f_1, X_1, \dots, f_n, X_n) &\mapsto f_n \circ \dots \circ f_1: X_0 \rightarrow X_n \\ (X) &\mapsto \text{Id}_X \end{aligned}$$

Exercise 35. Check that η and ϵ are natural transformations.

Now, given a graph G , compute the composition

$$F(G) \xrightarrow{F(\eta_G)} F \circ U \circ F(G) \xrightarrow{\epsilon_{F(G)}} F(G)$$

It sends an object X to itself, and for $(X_0, f_1, X_1, \dots, f_n, X_n)$, we compute

$$\begin{aligned} \epsilon_{F(G)}(F(\eta_G)(X_0, f_1, X_1, \dots, f_n, X_n)) \\ &= \epsilon_{F(G)}(X_0, (X_0, f_1, X_1), X_1, \dots, (X_{n-1}, f_n, X_n), X_n) \\ &= (X_{n-1}, f_n, X_n) \circ \dots \circ (X_0, f_1, X_1) \\ &= (X_0, f_1, X_1, \dots, f_n, X_n) \end{aligned}$$

Hence, we have the first required equality. For the second, let $\mathcal{C}$ be a small category. We need to compute the composition

$$U(\mathcal{C}) \xrightarrow{\eta_{U(\mathcal{C})}} U \circ F \circ U(\mathcal{C}) \xrightarrow{U(\epsilon_{\mathcal{C}})} U(\mathcal{C})$$

It sends a vertex X to itself. Now, for f an arrow in $A(U(\mathcal{C}))$, we compute

$$\begin{aligned} (U(\epsilon_{\mathcal{C}}))_A((\eta_{U(\mathcal{C})})_A(f)) &= (U(\epsilon_{\mathcal{C}}))_A(s_{U(\mathcal{C})}(f), f, t_{U(\mathcal{C})}(f)) \\ &= f \end{aligned}$$

This gives the second desired equality, hence we have exhibited an adjunction $F \dashv U$, between the category of small categories and the category of directed graphs.

Exercise 36. Compute the bijection associated to the adjunction $F \dashv U$,

$$\theta_{G,\mathcal{C}}: \text{Hom}_{\underline{\text{Graph}}}(G, U(\mathcal{C})) \simeq \text{Hom}_{\underline{\text{Cat}}}(F(G), \mathcal{C}).$$

3.3 Unicity of adjoints

Proposition 3.3.1. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor, and $G, G': \mathcal{D} \rightarrow \mathcal{C}$ be two functors. Assume that both G and G' are right adjoints to F , (i.e that $F \dashv G$ and $F \dashv G'$ are two adjoint pairs). Then, there exists a natural isomorphism $G \simeq G'$.*

The proof of the above proposition will use the following lemma. This would usually be considered a corollary of the Yoneda lemma (in Chapter 5). However, with the particular foundations we have chosen for this class, the Yoneda lemma will not be strong enough to imply such a statement for non-small categories.

Lemma 3.3.2. *Let $\mathcal{C}$ be a category, and Z, Z' be two objects of $\mathcal{C}$, such that there exists a natural isomorphism*

$$\beta: \text{Hom}_{\mathcal{C}}(-, Z) \simeq \text{Hom}_{\mathcal{C}}(-, Z')$$

Then, $\beta_Z(\text{Id}_Z): Z \rightarrow Z'$ is an isomorphism in $\mathcal{C}$, with inverse $(\beta_{Z'})^{-1}(\text{Id}_{Z'}): Z' \rightarrow Z$.

Proof. Let $f = \beta_Z(\text{Id}_Z): Z \rightarrow Z'$, and $g = (\beta_{Z'})^{-1}(\text{Id}_{Z'}): Z' \rightarrow Z$. Consider the following commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(Z', Z) & \xrightarrow{\beta_{Z'}} & \text{Hom}_{\mathcal{C}}(Z', Z') \\ \text{Hom}_{\mathcal{C}}(f, Z) \downarrow & & \downarrow \text{Hom}_{\mathcal{C}}(f, Z') \\ \text{Hom}_{\mathcal{C}}(Z, Z) & \xrightarrow{\beta_Z} & \text{Hom}_{\mathcal{C}}(Z, Z') \end{array}$$

Starting with g in the top left corner, we get $\text{Id}_{Z'}$ in the top right corner, and f in the bottom right corner. On the other hand, we get $g \circ f$ in the bottom left corner, hence $\beta_Z(g \circ f) = f = \beta_Z(\text{Id}_Z)$. Since β_Z is a bijection, we must have $g \circ f = \text{Id}_Z$. For the other composition, consider the commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(Z, Z) & \xrightarrow{\beta_Z} & \text{Hom}_{\mathcal{C}}(Z, Z') \\ \text{Hom}_{\mathcal{C}}(g, Z) \downarrow & & \downarrow \text{Hom}_{\mathcal{C}}(g, Z') \\ \text{Hom}_{\mathcal{C}}(Z', Z) & \xrightarrow{\beta_{Z'}} & \text{Hom}_{\mathcal{C}}(Z', Z') \end{array}$$

Once again, starting with Id_Z in the top left corner and going around the diagram gives the equality $f \circ g = \text{Id}_{Z'}$. This proves that $f: Z \rightarrow Z'$ and $g: Z' \rightarrow Z$ are inverse isomorphisms to each other. $\square$

Proof of Proposition 3.3.1. Let X be an object of $\mathcal{C}$ and Y an object of $\mathcal{D}$. Consider $\theta_{X,Y}: \text{Hom}_{\mathcal{C}}(X, G(Y)) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), Y)$ and $\theta'_{X,Y}: \text{Hom}_{\mathcal{C}}(X, G'(Y)) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), Y)$ the two natural isomorphisms associated to the two adjunctions, and define

$$\varphi_{X,Y} = (\theta'_{X,Y})^{-1} \circ \theta_{X,Y}: \text{Hom}_{\mathcal{C}}(X, G(Y)) \simeq \text{Hom}_{\mathcal{C}}(X, G'(Y))$$

By Lemma 3.3.2, $\alpha_Y = \varphi_{G(Y),Y}(\text{Id}_{G(Y)}): G(Y) \rightarrow G'(Y)$ is an isomorphism for all Y . Hence, it is sufficient to show that α is a natural transformation. Let $f: Y \rightarrow Z$ be a map in $\mathcal{D}$, we need to show that the following diagram is commutative

$$\begin{array}{ccc} G(Y) & \xrightarrow{\alpha_Y} & G'(Y) \\ G(f) \downarrow & & \downarrow G'(f) \\ G(Z) & \xrightarrow{\alpha_Z} & G'(Z) \end{array} \quad (3.4)$$

Hence, we need to show that $G'(f) \circ \alpha_Y = G(f) \circ \alpha_Z$. First, consider the commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(G(Y), G(Y)) & \xrightarrow{\varphi_{G(Y),Y}} & \text{Hom}_{\mathcal{C}}(G(Y), G'(Y)) \\ \text{Hom}_{\mathcal{C}}(G(Y), G(f)) \downarrow & & \downarrow \text{Hom}_{\mathcal{C}}(G(Y), G'(f)) \\ \text{Hom}_{\mathcal{C}}(G(Y), G(Z)) & \xrightarrow{\varphi_{G(Y),Z}} & \text{Hom}_{\mathcal{C}}(G(Y), G'(Z)) \end{array}$$

Starting with $\text{Id}_{G(Y)}$ in the top left corner gives the equality. $G'(f) \circ \alpha_Y = \varphi_{G(Y),Z}(G(f))$. Now, it remains to be proven that $\varphi_{G(Y),Z}(G(f)) = \alpha_Z \circ G(f)$. To do so, consider the commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(G(Z), G(Z)) & \xrightarrow{\varphi_{G(Z),Z}} & \text{Hom}_{\mathcal{C}}(G(Z), G'(Z)) \\ \text{Hom}_{\mathcal{C}}(G(f), G(Z)) \downarrow & & \downarrow \text{Hom}_{\mathcal{C}}(G(f), G'(Z)) \\ \text{Hom}_{\mathcal{C}}(G(Y), G(Z)) & \xrightarrow{\varphi_{G(Y),Z}} & \text{Hom}_{\mathcal{C}}(G(Y), G'(Z)) \end{array}$$

Starting with $\text{Id}_{G(Z)}$ in the top left corner gives the desired equality, $G(f) \circ \alpha_Z = \varphi_{G(Y),Z}$, which concludes the proof. $\square$

3.4 A useful property of adjunctions

Lemma 3.4.1. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ be a pair of adjoint functors $F \dashv G$, with the natural isomorphism*

$$\theta_{X,Y}: \text{Hom}_{\mathcal{C}}(X, G(Y)) \simeq \text{Hom}_{\mathcal{D}}(F(X), Y).$$

Let X and X' be objects of $\mathcal{C}$, and Y and Y' be objects of $\mathcal{D}$.

- if $f: X \rightarrow X'$, $g: X \rightarrow G(Y)$ and $g': X' \rightarrow G(Y)$ are three maps in $\mathcal{C}$. Then, the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ & \searrow g & \swarrow g' \\ & G(Y) & \end{array} \quad (3.5)$$

is commutative if and only if the following diagram is commutative.

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(X') \\ & \searrow \theta_{X,Y}(g) & \swarrow \theta_{X',Y}(g') \\ & Y & \end{array} \quad (3.6)$$

- Dually, if $h: Y \rightarrow Y'$, $k: F(X) \rightarrow Y$ and $k': F(X) \rightarrow Y'$ are maps in $\mathcal{D}$, then the diagram

$$\begin{array}{ccc} & F(X) & \\ k \swarrow & & \searrow k' \\ Y & \xrightarrow{h} & Y' \end{array}$$

is commutative if and only if the following diagram is commutative

$$\begin{array}{ccc} & X & \\ \theta_{X,Y}^{-1}(k) \swarrow & & \searrow \theta_{X,Y}^{-1}(k') \\ G(Y) & \xrightarrow{G(h)} & G(Y') \end{array}$$

Proof. Let $f: X \rightarrow X'$, $g: X \rightarrow G(Y)$ and $g': X' \rightarrow G(Y)$ be maps in $\mathcal{C}$. Use the naturality of θ to get the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X', G(Y)) & \xrightarrow{\theta_{X',Y}} & \mathrm{Hom}_{\mathcal{D}}(F(X'), Y) \\ \mathrm{Hom}_{\mathcal{C}}(f, G(Y)) \downarrow & & \downarrow \mathrm{Hom}_{\mathcal{D}}(F(f), Y) \\ \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) & \xrightarrow{\theta_{X,Y}} & \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \end{array}$$

Now, starting with $g': X' \rightarrow G(Y)$ in the top left corner, the commutativity of the diagram gives the equality $\theta_{X,Y}(g' \circ f) = \theta_{X',Y}(g') \circ F(f)$. We now see that the commutativity of diagram (3.6) is equivalent to the equality $\theta_{X,Y}(g' \circ f) = \theta_{X,Y}(g)$, and since $\theta_{X,Y}$ is a bijection this is equivalent to the equality $g' \circ f = g$, hence to the commutativity of diagram (3.5). This proves the first claim of the lemma.

Exercise 37. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ be an adjoint pair, $F \dashv G$. Consider $F^{\mathrm{op}}: \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{D}^{\mathrm{op}}$ and $G^{\mathrm{op}}: \mathcal{D}^{\mathrm{op}} \rightarrow \mathcal{C}^{\mathrm{op}}$ defined as in exercise 14.

- Show that we have an adjunction $G^{\text{op}} \dashv F^{\text{op}}$ (observe that left and right adjoints are exchanged).
- Deduce the second claim of the lemma by reasoning with opposite categories.

□

Exercise 38. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$, be a pair of adjoint functors $F \dashv G$. For each of the following diagrams in $\mathcal{C}$, find diagrams in $\mathcal{D}$ which commute if and only if the initial diagram commutes. There may be more than one correct answer in some cases.

$$\begin{array}{ccc}
 X & \xrightarrow{f} & G(Y) \\
 h \downarrow & & \downarrow G(g) \\
 X' & \xrightarrow{f'} & G(Y')
 \end{array} \tag{3.7}$$

$$\begin{array}{ccc}
 X_0 & \xrightarrow{f_1} & X_1 \\
 f_2 \downarrow & & \downarrow g_1 \\
 X_2 & \xrightarrow{g_2} & G(Y)
 \end{array} \tag{3.8}$$

$$\begin{array}{ccc}
 X & \xrightarrow{f_1} & G(Y_1) \\
 f_2 \downarrow & & \downarrow G(g_1) \\
 G(Y_2) & \xrightarrow{G(g_2)} & G(Y_0)
 \end{array} \tag{3.9}$$

$$\begin{array}{ccccccc}
 X_0 & \xrightarrow{f_2} & X_2 & \xrightarrow{f_3} & X_3 & \xrightarrow{g'_3} & G(Y_1) \\
 f_1 \downarrow & & \downarrow g_2 & & \downarrow g_3 & & \downarrow G(h_1) \\
 X_1 & \xrightarrow{g_1} & G(Y_3) & \xrightarrow{h_3} & G(Y_2) & \xrightarrow{h_2} & G(Y_0)
 \end{array} \tag{3.10}$$

Chapter 4

Limits and colimits

4.1 The category of cones

4.1.1 Diagrams

Definition 4.1.1. Let $\mathcal{C}$ be a category. A **diagram** in $\mathcal{C}$ is the data of

- a small category $\mathcal{I}$,
- a functor $D: \mathcal{I} \rightarrow \mathcal{C}$

When the small category $\mathcal{I}$ is fixed, we may refer to those diagrams as $\mathcal{I}$ -diagrams. We will say that $\mathcal{I}$ is the **shape** of the diagram.

Example 4.1.2. Let us list a few examples of small categories $\mathcal{I}$, that will frequently appear as shapes of diagrams. We will use the functor from Example 3.2.4 to specify the small category $\mathcal{I}$ through graphs, and then taking the free category.

- The first class of examples is that of **discrete categories**. Those are categories with some set of objects and **only** identity morphisms.
 - The most common example will be the category with two objects, and only identity morphisms.
 - We may also consider a category with n objects, and only identity morphisms, or even some set of object with infinite cardinality.
 - A fundamental example is also $\mathcal{I}$ being the **empty** category.
- Then, a lot of examples correspond to particular partially ordered sets seen as categories :

$$\begin{array}{ccc} & & \cdot \\ & & \downarrow \\ \cdot & \longrightarrow & \cdot \end{array} \quad (4.1)$$

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \downarrow & & \\ \cdot & & \end{array} \quad (4.2)$$

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \dots \quad (4.3)$$

$$\dots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \quad (4.4)$$

$$\dots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \dots \quad (4.5)$$

Where the last three examples should be interpreted as $\mathbb{N}$, $\mathbb{N}^{\text{op}}$ and $\mathbb{Z}$

- an important example, which does not fit in the two previous categories is the following :

$$\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ & \curvearrowright & \\ \cdot & \xleftarrow{\quad} & \cdot \end{array} \quad (4.6)$$

- Finally, later on, we will see examples where $\mathcal{I}$ is some abstract small category.

Remark 4.1.3. Diagrams are simply functors from a small category. Given a category $\mathcal{C}$ and some fixed shape $\mathcal{I}$, we can consider the category of functors $\text{Fun}(\mathcal{I}, \mathcal{C})$ (see Definition 2.2.6). The difference between functors and diagrams is simply one of **point of view**. When considering a diagram, we usually do not care about any interpretation of the source category $\mathcal{I}$, it is simply a model for the shape of the diagram. In particular, it will be extremely frequent that we skip the description of the functor altogether and just recall the image of the functor $D: \mathcal{I} \rightarrow \mathcal{C}$. For example, if $\mathcal{I}$ is the category corresponding to (4.1) to specify some diagram $D: \mathcal{I} \rightarrow \mathcal{C}$ in some category $\mathcal{C}$, we will just specify its image in $\mathcal{C}$ as follows:

$$\begin{array}{ccc} & X & \\ & \downarrow f & \\ Y & \xrightarrow{g} & Z \end{array}$$

Where X, Y and Z are objects of $\mathcal{C}$, and f and g are morphisms in $\mathcal{C}$. In this section, we will write $\mathcal{C}^{\mathcal{I}}$ instead of $\text{Fun}(\mathcal{I}, \mathcal{C})$, and it should be understood as a reminder that we are thinking about the objects in this category as **diagrams**.

4.1.2 Cones (and cocones)

Definition 4.1.4. Let $\mathcal{C}$ be a category and $\mathcal{I}$ a small category. Define $\Delta: \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$ as follows

$$\Delta: \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$$

$$X \mapsto \begin{cases} \Delta X: \mathcal{I} & \rightarrow \mathcal{C} \\ i & \mapsto X \\ i \rightarrow j & \mapsto \text{Id}_X: X \rightarrow X \end{cases}$$

Given an object X in $\mathcal{C}$, ΔX is called the **constant diagram** with value X . Given a morphism $f: X \rightarrow Y$ in $\mathcal{C}$, define $\Delta f: \Delta X \rightarrow \Delta Y$ as the natural transformation such that $(\Delta f)_i = f: X \rightarrow Y$ for all objects i of $\mathcal{I}$.

Exercise 39. Check that Δ is indeed a functor.

Definition 4.1.5. Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram in some category $\mathcal{C}$. A **cone** on D is the data of

- an object X of $\mathcal{C}$
- a natural transformation $\alpha: \Delta X \rightarrow D$

Example 4.1.6. Take $\mathcal{I}$ to be the discrete category with two objects, i and j and no non-identity morphisms. Then, a diagram $D: \mathcal{I} \rightarrow \underline{\text{Set}}$ is simply the data of two sets $X = D(i)$ and $Y = D(j)$. A cone on D is then exactly the data of some set Z , and two maps

$$\begin{array}{ccc} & Z & \\ f \swarrow & & \searrow g \\ X & & Y \end{array}$$

Remark 4.1.7. Starting with the definition of a cone, one can dualize every definition (and statement, and proof, and exercise) in this chapter. Doing so will give rise to dual notions, usually denoted with a "co" prefix. We will focus on one half of the notions for the first few sections but it is a good exercise to work through the dual point of view while reading along.

Exercise 40. Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram in some category $\mathcal{C}$, and X be an object of $\mathcal{C}$. Show that the data of a cone $\alpha: \Delta X \rightarrow D$ is equivalent to the data of maps $\alpha_i: X \rightarrow D(i)$, for all objects $i \in \mathcal{I}$, such that for all maps $\varphi: i \rightarrow j$ in $\mathcal{I}$, the following triangle commutes

$$\begin{array}{ccc} & X & \\ \alpha_i \swarrow & & \searrow \alpha_j \\ D(i) & \xrightarrow{D(\varphi)} & D(j) \end{array}$$

Definition 4.1.8. Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram. Define the category of cones on D , $\underline{\text{Cone}}(D)$ to be the category with

- Objects are cones $(X, \alpha: \Delta X \rightarrow D)$
- Given two cones (X, α) and (Y, β) , morphisms are the morphisms $f: X \rightarrow Y$ in $\mathcal{C}$ such that the following diagram is commutative in $\mathcal{C}^{\mathcal{I}}$

$$\begin{array}{ccc} \Delta X & \xrightarrow{\Delta f} & \Delta Y \\ & \searrow \alpha & \swarrow \beta \\ & D & \end{array}$$

In particular, $\text{Hom}_{\underline{\text{Cone}}(D)}((X, \alpha), (Y, \beta))$ is defined as a subset of $\text{Hom}_{\mathcal{C}}(X, Y)$

- Identity and compositions are defined by restricting those of $\mathcal{C}$.

Example 4.1.9. Consider the degenerate example where $\mathcal{I} = \emptyset$ is the empty category (with no objects and no morphisms). Then, $\mathcal{C}^{\mathcal{I}}$ is the category with a single object, the empty functor, $D: \emptyset \rightarrow \mathcal{C}$. We still have a functor $\Delta: \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$, which sends every object to the empty functor, D , and every morphism to Id_D . But now, observe that a cone on D is a pair $(X, \alpha: \Delta X \rightarrow D)$, with $\Delta X = D$, and the only possible choice for α is Id_D . Thus, in that case, $\underline{\text{Cone}}(D)$ is nothing more than the category $\mathcal{C}$ itself (up to renaming the objects).

Example 4.1.10. A slightly less degenerate example is the case where $\mathcal{I}$ is the discrete category with a single object, which we will denote $\star$ (and only the identity morphism). In that case, functors from $\mathcal{I}$ to $\mathcal{C}$ are given by the data of a single object of $\mathcal{C}$, and we see that $\mathcal{C}^{\mathcal{I}}$ is exactly the category $\mathcal{C}$ itself. Now, pick some functor $D: \mathcal{I} \rightarrow \mathcal{C}$. This amounts to fixing some object X of $\mathcal{C}$ such that $D(\star) = X$. Now, a cone on D is the data of some object Y in $\mathcal{C}$, together with a map $f: Y \rightarrow X$. And the category $\underline{\text{Cone}}(D)$ has, for objects pairs (Y, f) , where Y is an object of $\mathcal{C}$ and $f: Y \rightarrow X$ is a map in $\mathcal{C}$, and for morphisms between (Y, f) and (Z, g) , morphisms $h: Y \rightarrow Z$ such that the following triangle is commutative in $\mathcal{C}$

$$\begin{array}{ccc} Y & \xrightarrow{h} & Z \\ & \searrow f & \swarrow g \\ & X & \end{array}$$

This category is known as the **slice category** of $\mathcal{C}$ over X , denoted $\mathcal{C}/X$ or $\mathcal{C} \downarrow X$ (also sometimes called the overcategory).

4.2 Limits

4.2.1 Initial and terminal objects

Definition 4.2.1. Let $\mathcal{C}$ be a category, and X be an object of $\mathcal{C}$

- the object X is **terminal** in $\mathcal{C}$ (or is a terminal object of $\mathcal{C}$) if, for any object Y of $\mathcal{C}$, the set $\text{Hom}_{\mathcal{C}}(Y, X)$ is a singleton.
- The object X is **initial** in $\mathcal{C}$ (or is an initial object of $\mathcal{C}$) if, for any object Y of $\mathcal{C}$, the set $\text{Hom}_{\mathcal{C}}(X, Y)$ is a singleton.

Remark 4.2.2. Observe that the definitions of terminal and initial objects are dual to one another. This means that we need only prove properties of terminal objects to deduce the dual properties of initial objects, see Remark 4.1.7.

Proposition 4.2.3. Let $\mathcal{C}$ be a category, and X and X' be two objects of $\mathcal{C}$. Assume that both X and X' are terminal objects, then there exists a unique isomorphism in $\mathcal{C}$ between X and X' .

Proof. Since X is assumed to be terminal, there exists a unique map $f: X' \rightarrow X$. Similarly, terminality of X' gives a unique map $g: X \rightarrow X'$. But now, the composition $f \circ g: X \rightarrow X$ gives a map in $\text{Hom}_{\mathcal{C}}(X, X)$ which must both contain

Id_X and be a singleton. Hence, $f \circ g = \text{Id}_X$ and similarly $g \circ f = \text{Id}_{X'}$. This proves that X and X' are uniquely isomorphic. $\square$

Remark 4.2.4. Because of the above proposition, we will often speak of **the** terminal object instead of **a** terminal object, since we do not need to distinguish between *uniquely isomorphic* objects. Note that the unicity is of importance. If X and Y are two objects of some category $\mathcal{C}$, such that there exists several distinct isomorphism between X and Y , say $f: X \rightarrow Y$ and $g: X \rightarrow Y$, then caution is needed when identifying them. Say for example that we want to say that for any Z in $\mathcal{C}$, $\text{Hom}_{\mathcal{C}}(Z, X)$ and $\text{Hom}_{\mathcal{C}}(Z, Y)$ are "the same set". It is true that those two sets will necessarily be in bijection. And such bijections can be obtained by post-composing by either f or g . But crucially, in general, post-composing by f or by g will give *distinct* bijections. To see this, pick $Z = X$ and consider the image of Id_X . Thus, one can only identify X and Y *through some specific isomorphism*. On the other hand, in the presence of a *unique* isomorphism, no such ambiguity is possible, since the isomorphism through which we make the identification is necessarily the only one that exists.

Exercise 41. Identify the initial and terminal objects (when they exist) in the categories of Example 1.1.4.

4.2.2 Limits and universal properties

Definition 4.2.5. Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram. The limit of D , if it exists, is the terminal object of $\underline{\text{Cone}}(D)$.

Remark 4.2.6. We define **the** limit, where **a** limit would be a more accurate term. However, as we have seen in remark 4.2.4, the limit is well defined up to unique isomorphism (in $\underline{\text{Cone}}(D)$).

Remark 4.2.7. We just defined the limit of a diagram, D as an object of $\underline{\text{Cone}}(D)$, that is, a pair $(X, \alpha: \Delta X \rightarrow D)$. In most cases, we will be mainly interested in the object of $\mathcal{C}$, X , and may refer to X itself as $\lim_{\leftarrow} D$ or the limit of D . In this context, we will usually write $X \simeq \lim_{\leftarrow} D$. In that case, we may speak of the **limit cone** to designate the pair (X, α) . Note however that when speaking in those terms, as an object of $\mathcal{C}$, the limit of D is no longer unique up to *unique* isomorphism, but only up to not-necessarily-unique isomorphism. This is why, whenever ambiguity is possible, the entire limit cone should be recalled.

The previous definition may be somewhat difficult to parse. One may also think about limits in terms of universal properties. From this point of view, we may rewrite the definition into an equivalent, but more enlightening form.

Alternative Definition 4.2.8. Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram. A **limit cone** is the data of

- some object X in $\mathcal{C}$,
- for all $i \in \text{obj}(\mathcal{I})$, maps $\alpha_i: X \rightarrow D(i)$

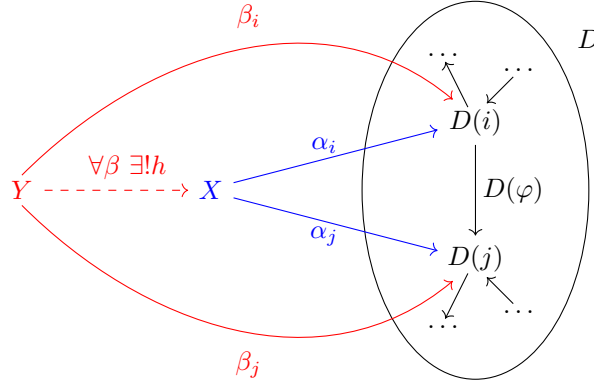


Figure 4.1: The universal property of a limit cone

Such that, for any map $\varphi: i \rightarrow j$ in $\mathcal{I}$, the following diagram commutes

$$\begin{array}{ccc}
 & D(i) & \\
 \alpha_i \nearrow & & \downarrow D(\varphi) \\
 X & & \\
 \alpha_j \searrow & & \downarrow \\
 & D(j) &
 \end{array}$$

And such that for any other cone $(Y, (\beta_i)_{i \in \text{obj}(\mathcal{I})})$, satisfying the same condition, there exists a unique map $h: Y \rightarrow X$ such that for all $i \in \text{obj}(\mathcal{I})$, $\beta_i = \alpha_i \circ h$.

Remark 4.2.9. The above rewriting of the definition correspond to the way we usually think about universal properties. We have some data, in the form of the diagram D , and we want to understand how to map into that data (i.e. the cones). Now, the limit cone, can be thought of as the best single-object approximation of that data, in that any other way of mapping some object into the diagram must factor through the limit cone. The situation is summarized in Figure 4.1. In black is the diagram D , or rather, its image in $\mathcal{C}$. In blue is (part of) the limit cone. In red is another cone. And the dashed red arrow, is the unique factorization of the red cone through the blue limit-cone. Note that every triangle in this diagram is assumed to commute.

Exercise 42. Convince yourself that Definitions 4.2.5 and 4.2.8 are equivalent. See Exercise 40.

Exercise 43. Let $\mathcal{I}$ be the discrete category with two objects, i and j , and no non-identity morphisms. Let X and Y be two sets, and $D: \mathcal{I} \rightarrow \mathbf{Set}$ be the functor sending i to X and j to Y . Using the arguments in Motivation 1.1.1, show that $X \times Y$, together with $\text{pr}_X: X \times Y \rightarrow X$ and $\text{pr}_Y: X \times Y \rightarrow Y$ is a limit cone. Prove the claim of Exercise 1 with the tools of this chapter.

4.3 Limits in practice

4.3.1 Computations in $\mathbf{Set}$

Let us look at a few examples of diagrams and their limits in $\mathbf{Set}$.

Example 4.3.1 (The empty limit). First, consider $\mathcal{I} = \emptyset$ the empty category, and the unique functor $D: \mathcal{I} \rightarrow \mathbf{Set}$. Looking back at Example 4.1.9, we see that the category $\mathbf{Cone}(D)$ is simply $\mathbf{Set}$ in disguise. Thus, the limit of D is the data of a terminal object in $\mathbf{Set}$. A terminal object in $\mathbf{Set}$ is simply a singleton, $X = \{*\}$, since, in that case, for any set Y , we clearly have that $\mathbf{Hom}_{\mathbf{Set}}(Y, X)$ is a singleton. Indeed, there is always a single map from a set Y to a singleton $\{*\}$, the map sending every element of Y to $*$.

Example 4.3.2 (Pairwise products). Now, in Exercise 43, we saw that if $\mathcal{I}$ is the discrete category with two objects 1 and 2, and $D: \mathcal{I} \rightarrow \mathbf{Set}$ is any functor with values $D(1) = X$ and $D(2) = Y$, then the limit cone is simply the product $X \times Y$ with its two projections, $\text{pr}_1: X \times Y \rightarrow X$ and $\text{pr}_2: X \times Y \rightarrow Y$.

Exercise 44. Use the universal property of the (pairwise) product to recover the diagonal map $\delta: X \rightarrow X \times X$ sending $x \in X$ to the pair (x, x) .

Example 4.3.3 (Arbitrary products). More generally, assume that $\mathcal{I}$ is any discrete category. That is, $\text{obj}(\mathcal{I})$ is an arbitrary set, and $\mathcal{I}$ does not admit any non-identity morphisms. We will identify $\mathcal{I}$ the category, with $\text{obj}(\mathcal{I})$ the set, for convenience. Fix some functor $D: \mathcal{I} \rightarrow \mathbf{Set}$. We will show that the limit of D is the (possibly infinite) product $\lim_{\leftarrow} D \simeq \prod_{i \in \mathcal{I}} D(i)$. First, we indeed have a cone by considering, for $i \in \mathcal{I}$ the projection

$$\text{pr}_i: \prod_{j \in \mathcal{I}} D(j) \rightarrow D(i)$$

There are no commutativity conditions to check, since there are no non-identity maps in $\mathcal{I}$. Now, given some other cone on D , i.e. some set X and for all $i \in \mathcal{I}$, maps $f_i: X \rightarrow D(i)$, observe that we may define a map

$$\begin{aligned} \prod_{i \in \mathcal{I}} f_i: X &\rightarrow \prod_{i \in \mathcal{I}} D(i) \\ x &\mapsto (f_i(x))_{i \in \mathcal{I}} \end{aligned}$$

Furthermore, this map satisfies the equality

$$\text{pr}_j \circ \prod_{i \in \mathcal{I}} f_i = f_j$$

by construction. Let us show that it is the unique map with such a property. Let $g: X \rightarrow \prod_{i \in \mathcal{I}} D(i)$ be a map such that the equality

$$\text{pr}_j \circ g = f_j$$

holds for all $j \in \mathcal{I}$. Then, for any $x \in X$, $g(x)$ must be of the form $(g(x)_i)_{i \in \mathcal{I}}$, by definition of the product. And the above equality then gives $g(x)_i = f_i(x)$ for all $i \in \mathcal{I}$. This then proves that $g(x) = (\prod_{i \in \mathcal{I}} f_i)(x)$, by definition of the product. Since the equality holds for all $x \in X$, we have $g = \prod_{i \in \mathcal{I}} f_i$. This proves the unicity of the map $\prod_{i \in \mathcal{I}} f_i$, hence we have shown that $\prod_{i \in \mathcal{I}} D(i)$, together with the projections pr_j , for $j \in \mathcal{I}$ form a limit cone for the diagram D .

Remark 4.3.4. Observe that Examples 4.3.1 and 4.3.2 are both particular cases of Example 4.3.3. For pairwise products, this is fairly transparent, but for the case of the empty limit, it should be interpreted as the product of a family of sets indexed by the empty set, also referred to as the empty product.

Example 4.3.5 (Equalizers). Let $\mathcal{I}$ be the category with two objects 1 and 2, and two parallel maps from 1 to 2, $\varphi: 1 \rightarrow 2$ and $\psi: 1 \rightarrow 2$, in addition to the two identities Id_1 and Id_2 . A diagram $D: \mathcal{I} \rightarrow \underline{\text{Set}}$ then amounts to the data of

- two sets, $X = D(1)$, $Y = D(2)$,
- two parallel maps $f = D(\varphi): X \rightarrow Y$ and $g = D(\psi): X \rightarrow Y$.

Now, a cone on D is the data of some set Z and two maps $\alpha_X: Z \rightarrow X$ and $\alpha_Y: Z \rightarrow Y$ such that

$$f \circ \alpha_X = \alpha_Y = g \circ \alpha_X. \quad (4.7)$$

To understand the above condition, consider the following diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\alpha_X} & X & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & Y \\ & & & \searrow \alpha_Y & \\ & & & & \end{array}$$

To have a cone, the triangles obtained by discarding either f or g must both be commutative. In particular, observe that if we already know that Z , together with α_X and α_Y is a cone, then the data of α_Y becomes entirely redundant, since it can be recovered through Equation (4.7). For this reason, when thinking about cones over this particular diagram, we will only recall the map α_X , together with the condition that

$$f \circ \alpha_X = g \circ \alpha_X \quad (4.8)$$

since this is equivalent to the existence of a unique $\alpha_Y: Z \rightarrow Y$ making Z , α_X and α_Y into a cone in the usual sense.

Now, we claim that a limit cone of D is given by a particular subset $A \subset X$, together with its inclusion, where the subset is defined via

$$A = \{x \in X \mid f(x) = g(x)\}$$

Indeed, the inclusion $i_A: A \rightarrow X$ satisfies the equation

$$f \circ i_A = g \circ i_A$$

by construction. To see that it is universal, let Z be some set, together with a map $\alpha: Z \rightarrow X$ such that (4.8) is verified. Then, given any $z \in Z$, we have that $f(\alpha(z)) = g(\alpha(z))$, hence $\alpha(z) \in A$. This means that α factors through the inclusion $i_A: A \hookrightarrow X$. Since i_A is an inclusion, this factorization is necessarily unique, hence $\lim_{\leftarrow} D$ is the cone (A, i_A) . The limit of this particular shape of diagram is of fundamental interest, hence has been given the name **equalizer**, and will often be denoted $\text{Eq}(f, g)$. Depending on context, $\text{Eq}(f, g)$ may stand for the set A itself, or for the limit cone $(A, A \rightarrow X)$.

Example 4.3.6 (Pullbacks). Let $\mathcal{I}$ be the category with three objects 0, 1 and 2 and two non-identity maps $\varphi: 0 \rightarrow 2$ and $\psi: 1 \rightarrow 2$. A diagram $D: \mathcal{I} \rightarrow \underline{\text{Set}}$ then correspond to the following data :

$$\begin{array}{ccc} & X = D(0) & \\ & \downarrow f = D(\varphi) & \\ Y = D(1) & \xrightarrow{g = D(\psi)} & Z = D(2) \end{array}$$

By exactly the same reasoning as in Example 4.3.5, in order to specify a cone on D , it is sufficient to specify a set W and two maps $\alpha_X: W \rightarrow X$ and $\alpha_Y: W \rightarrow Y$ such that

$$f \circ \alpha_X = g \circ \alpha_Y \quad (4.9)$$

Now, observe that the product $X \times Y$ comes with two maps $\text{pr}_1: X \times Y \rightarrow X$ and $\text{pr}_2: X \times Y \rightarrow Y$. However, pr_1 and pr_2 do not satisfy equation (4.9). Nevertheless, we have two parallel maps

$$\begin{array}{ccccc} & & X & & \\ & \nearrow \text{pr}_1 & & \searrow f & \\ X \times Y & & & & Z \\ & \searrow \text{pr}_2 & & \nearrow g & \\ & & Y & & \end{array}$$

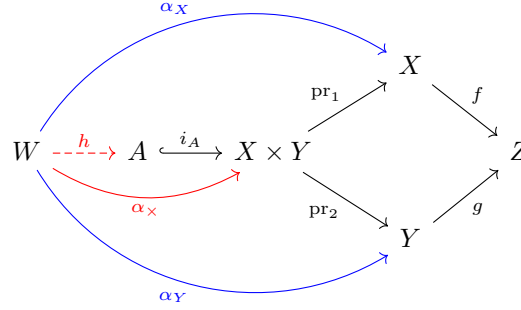
Now, we can compute the equalizer $\text{Eq}(f \circ \text{pr}_1, g \circ \text{pr}_2)$, which gives a map $i_A: A \rightarrow X \times Y$. By construction, $\text{pr}_1 \circ i_A: A \rightarrow X$ and $\text{pr}_2 \circ i_A: A \rightarrow Y$ satisfy Equation (4.9), and hence, A forms a cone with those two maps. Let us prove that it is a limit cone.

Let W be a set, and $\alpha_X: W \rightarrow X$ and $\alpha_Y: W \rightarrow Y$ be two maps such that Equation (4.9) is satisfied (i.e. a cone on D). By Example 4.3.2, the two maps α_X and α_Y must factor uniquely through $X \times Y$, via some map $\alpha_{\times}: W \rightarrow X \times Y$. But now, by assumption, we have

$$f \circ \text{pr}_1 \circ \alpha_{\times} = f \circ \alpha_X = g \circ \alpha_Y = g \circ \text{pr}_2 \circ \alpha_{\times}$$

This proves that $\alpha_{\times}$ satisfies Equation (4.7) (from Example 4.3.5) with respect to the diagram given by $f \circ \text{pr}_1: X \times Y \rightarrow Z$ and $g \circ \text{pr}_2: X \times Y \rightarrow Z$, hence

the map $\alpha_\times : W \rightarrow X \times Y$ must factor through A . The situation is summarized in the following diagram



The blue parts of the diagram is the data of an arbitrary cone on D . The black part has been constructed earlier, and we have shown that the blue cone must factor through the red solid map to $X \times Y$, which further factors through the dotted red arrow to A . Now, we need to prove that this dotted arrow is unique. That is, given some $h' : W \rightarrow A$ such that $\text{pr}_1 \circ i_A \circ h' = \alpha_X$ and $\text{pr}_2 \circ i_A \circ h' = \alpha_Y$, then we necessarily have $h' = h$. Let $w \in W$ and observe that $i_A \circ h'(w)$ correspond to some element $(x, y) \in X \times Y$. However, by construction of the product, we must have $\text{pr}_1(x, y) = x = \alpha_X(w)$, by assumption. And similarly for the second projection. Hence, $i_A \circ h'(w) = (\alpha_X(w), \alpha_Y(w)) = i_A \circ h(w)$. Since this is true for any $w \in W$, we must have $i_A \circ h = i_A \circ h'$. Now, since i_A is a monomorphism (see Exercise 8), we must have $h = h'$, which proves the unicity of h . Finally, we have shown that A , together with $\text{pr}_1 \circ i_A$ and $\text{pr}_2 \circ i_A$ is a limit cone for the diagram D .

It will be convenient to have a more explicit and direct description of the limit. Observe that the subset $A \subset X \times Y$ can be explicitly described as

$$A = \{(x, y) \mid x \in X, y \in Y, f(x) = g(y)\} \quad (4.10)$$

From this description, we can directly define the maps $\text{pr}_1 : A \rightarrow X$ and $\text{pr}_2 : A \rightarrow Y$ as those maps sending the pair (x, y) to $x \in X$ and $y \in Y$ respectively.

Exercise 45. Prove directly (without using Examples 4.3.5 or 4.3.2) that the set described in Equation (4.10), together with the map pr_1 and pr_2 forms a limit cone for the diagram D .

This particular shape of limits is perhaps, together with products, the most common limit appearing in category theory. They are named **pullbacks**, and we may say that " $\text{pr}_1 : A \rightarrow X$ is the pullback of $g : Y \rightarrow Z$ along $f : X \rightarrow Z$ ", and we will often write $X \times_Z Y$, or more precisely $X \times_{f,g} Y$, to stand for the limit A . Furthermore, whenever a square of the form :

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

is actually a limit cone, we will simply state that it is a **pullback square**, or that W is the pullback, which we may write $W \simeq X \times_Z Y$.

Exercise 46. Let X, Y and Z be three sets, and $f: X \rightarrow Z$, $g: Y \rightarrow Z$ be two maps. We aim to compute the pullback

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

in particular cases.

- Assume that $Y = \{y\}$ is a singleton with $g(y) = z$. Show that $X \times_Z Y \rightarrow X$ is the inclusion $f^{-1}(z) \hookrightarrow X$.
- Assume that both $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ are inclusions of subsets. Show that $X \times_Z Y = X \cap Y$ is the intersection, with $X \times_Z Y \rightarrow X$ and $X \times_Z Y \rightarrow Y$ the subset inclusions.
- Assume that $g: Y \rightarrow Z$ is the inclusion of a subset. Show that $X \times_Z Y \rightarrow X$ is the inclusion of $f^{-1}(Y) \rightarrow X$ and that the map $X \times_Z Y \rightarrow X$ is the restriction $f|_{f^{-1}(Y)}: f^{-1}(Y) \rightarrow Y$.
- Let $f_1, f_2: X \rightarrow Y$ be two maps between sets. Let Z be the product $Y \times Y$, and $f: X \rightarrow Z$ be defined by $f(x) = (f_1(x), f_2(x))$, for $x \in X$. Define $g = \delta: Y \rightarrow Y \times Y = Z$ as in Exercise 44. Prove that the diagram

$$\begin{array}{ccc} \text{Eq}(f_1, f_2) & \hookrightarrow & X \\ f_1=f_2 \downarrow & & \downarrow f \\ Y & \xrightarrow{\delta} & Y \times Y \end{array}$$

is a pullback square.

Exercise 47 (From Partial Exam of November 2023). Consider the following commutative diagram in Set.

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$

For each of those claims, give a proof or find a counter-example.

1. If the right and left squares (i.e. those given by g, g', β, γ and f, f', α, β respectively) are both pullback-squares, then the rectangle is a pullback square (i.e. the square given by $g \circ f, g' \circ f', \alpha, \gamma$).
2. If the right square and the rectangle are both pullback squares, then the left square is a pullback square.
3. If the left square and the rectangle are both pullback squares, then the right square is a pullback square.

4.3.2 Existence of limits and completeness

Definition 4.3.7. Let $\mathcal{C}$ be a category, and $\mathcal{I}$ some fixed small category. We say that $\mathcal{C}$ **admits all $\mathcal{I}$ -shaped limits** if, for all diagram $D: \mathcal{I} \rightarrow \mathcal{C}$, the limit $\lim_{\leftarrow} D$ exists. When $\mathcal{I}$ is a discrete category with two objects, we will say that $\mathcal{C}$ admits all pairwise products. If $\mathcal{I}$ is the category with two parallel maps, we will say that $\mathcal{C}$ admits all equalizers.

If $\mathcal{C}$ admits all $\mathcal{I}$ shaped limits for $\mathcal{I}$ any small discrete category, we will say that $\mathcal{C}$ admits all products.

If $\mathcal{C}$ admits all $\mathcal{I}$ -shaped limits for $\mathcal{I}$ any small category, we will say that $\mathcal{C}$ is **complete**.

Remark 4.3.8. Given a small category $\mathcal{I}$, it is entirely possible for some category $\mathcal{C}$ to admit *some* $\mathcal{I}$ shaped limits, but not *all* $\mathcal{I}$ -shaped limits. Explicitly, it may be the case that there exists two diagrams $D_1: \mathcal{I} \rightarrow \mathcal{C}$ and $D_2: \mathcal{I} \rightarrow \mathcal{C}$ such that D_1 admits a limit and D_2 does not. Understanding such an example is the object of the next exercise.

Exercise 48. Let $n > 0$ be an integer. Let $\mathcal{C}_n \subset \mathbf{Set}$ be the full subcategory whose objects are the sets $\{1, \dots, k\}$, for $0 \leq k \leq n$ (the set corresponding to $k = 0$ is the empty set). Let $\mathcal{I}$ be the discrete category with two objects, 1 and 2. Let $0 \leq k_1, k_2 \leq n$ be two integers and $D: \mathcal{I} \rightarrow \mathcal{C}$ be the diagram sending 1 to k_1 and 2 to k_2

- Show that if $k_1 \times k_2 \leq n$, then D admits a limit, and the limit is the set $\{1, \dots, k_1 \times k_2\}$. Determine a limit cone.
- Show that if $k_1 \times k_2 > n$, then D does not admit a limit.
- Deduce that $\mathcal{C}_n$ does not admit all pairwise products.
- Now, consider the full subcategory $\mathcal{C} \subset \mathbf{Set}$ whose objects are the sets $\{1, \dots, k\}$ for $k > 0$ an integer. Show that $\mathcal{C}$ admits all pairwise products.

Exercise 49. Let $\mathcal{C}$ be a category. Assume that $\mathcal{C}$ admits all pairwise products. Let $\mathcal{I}$ be a finite, discrete category, and $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram. Show that $\lim_{\leftarrow} D$ exists. In that case, we will say that $\mathcal{C}$ admits all **finite** products.

The following theorem is of immense practical use. A lot of the categories we will be interested in will be complete. And when they are not, we will usually want to relate them to complete categories. We need an efficient way to prove completeness of a category, and to compute limits in a complete category. This theorem provides us with a way to do both.

Theorem 4.3.9. Let $\mathcal{C}$ be a category. The following are equivalent.

- $\mathcal{C}$ is complete,
- $\mathcal{C}$ admits arbitrary products and equalizers,
- $\mathcal{C}$ admits arbitrary products and pullbacks.

Proof. Clearly, if a category is complete, then it admits arbitrary products, equalizers and pullbacks. Now, observe that in Example 4.3.6, we exhibited a computation of pullbacks from pairwise products and equalizers (in the category Set), and in Exercise 46, we exhibited the equalizer of two maps as a pullback.

Exercise 50. Generalizing Example 4.3.6 and Exercise 46 to an arbitrary category, show that the following assertions are equivalent

- $\mathcal{C}$ admits all finite products and pullbacks,
- $\mathcal{C}$ admits all finite products and equalizers.
- (To do after having read this proof) $\mathcal{C}$ admits all finite limits.

The above exercise implies the equivalence between the second and third assertions in the theorem. Thus, it is sufficient to prove that a category admitting all products and equalizers is complete. Assume that $\mathcal{C}$ is such a category, $\mathcal{I}$ a small category and $D: \mathcal{I} \rightarrow \mathcal{C}$ is some diagram. Let us explicitly compute a limit cone for D . We follow the point of view of Alternative Definition 1.2.3 for the small category $\mathcal{I}$. Let $I = \text{obj}(\mathcal{I})$ be the set of objects in $\mathcal{I}$, and $M = \text{mor}(\mathcal{I})$ be the set of morphisms of $\mathcal{I}$, recall that $s, t: M \rightarrow I$ associate to any map its source and target respectively. Since, by assumption, all products exist in $\mathcal{C}$, we may consider the products $\prod_{i \in I} D(i)$ and $\prod_{\varphi \in M} D(t(\varphi))$. We will now define two maps from the former to the latter. Let $\varphi \in M$ be a morphism in $\mathcal{I}$. Consider the composition

$$\prod_{i \in I} D(i) \xrightarrow{\text{pr}_{s(\varphi)}} D(s(\varphi)) \xrightarrow{D(\varphi)} D(t(\varphi)) \quad (4.11)$$

The compositions $D(\varphi) \circ \text{pr}_{s(\varphi)}$, for $\varphi \in M$ form a cone on the (discrete) diagram $M \rightarrow \mathcal{C}$, sending φ to $D(t(\varphi))$ (this is the diagram whose limit is $\prod_{\varphi \in M} D(t(\varphi))$). Hence, it factors through a unique map

$$D \circ \text{pr}_s: \prod_{i \in I} D(i) \rightarrow \prod_{\varphi \in M} D(t(\varphi)).$$

On the other hand, given $\varphi \in M$, we may simply consider the projection

$$\text{pr}_{t(\varphi)}: \prod_{i \in I} D(i) \rightarrow D(t(\varphi))$$

Similarly, this is a cone, and thus factors through a unique map

$$\text{pr}_t: \prod_{i \in I} D(i) \rightarrow \prod_{\varphi \in M} D(t(\varphi))$$

We now have two parallel maps (coming from two distinct cones)

$$L(D) = \text{Eq}(D \circ \text{pr}_s, \text{pr}_t) \dashrightarrow \prod_{i \in \mathcal{I}} D(i) \begin{array}{c} \xrightarrow{D \circ \text{pr}_s} \\ \xrightarrow{\text{pr}_t} \end{array} \prod_{\varphi \in \mathcal{I}} D(t(\varphi))$$

and we can take the equalizer. Now, consider the object $L(D)$ of $\mathcal{C}$, together with the morphisms $\beta_i = \text{pr}_i \circ \alpha: L(D) \rightarrow D(i)$, for $i \in I$. Let us show that it is a limit cone for D .

First, we need to show that it is a cone. Let $\varphi: i \rightarrow j$ be a morphism in $\mathcal{I}$, and consider the (non-commutative) diagram

$$\begin{array}{ccccc}
 & & \prod_{k \in I} D(k) & \xrightarrow{\text{pr}_t} & \prod_{\psi \in M} D(t(\psi)) \\
 & & \downarrow D \circ \text{pr}_s & \searrow & \downarrow \text{pr}_\varphi \\
 L(D) & \xrightarrow{\alpha} & \prod_{k \in I} D(k) & \xrightarrow{D(\varphi)} & \prod_{\psi \in M} D(t(\psi)) \\
 & \searrow \beta_i & \downarrow \text{pr}_{s(\varphi)} & & \downarrow \text{pr}_\varphi \\
 & & D(i) & \xrightarrow{D(\varphi)} & D(j) \\
 & \searrow \beta_j & & &
 \end{array}$$

We want to show that $\beta_j = D(\varphi) \circ \beta_i$. By definition, $\beta_j = \text{pr}_j \circ \alpha$, but by construction, we also have $\text{pr}_j = \text{pr}_{t(\varphi)} = \text{pr}_\varphi \circ \text{pr}_t$. This gives, $\beta_j = \text{pr}_\varphi \circ \text{pr}_t \circ \alpha$. Or, since $(L(D), \alpha)$ is the equalizer of pr_t and $D \circ \text{pr}_s$, we have $\text{pr}_t \circ \alpha = D \circ \text{pr}_s \circ \alpha$, which gives $\beta_j = \text{pr}_\varphi \circ D \circ \text{pr}_s \circ \alpha$. But, the composition $\text{pr}_\varphi \circ D \circ \text{pr}_s$ is precisely the composition defined in Equation 4.11, hence is equal to $D(\varphi) \circ \text{pr}_{s(\varphi)}$, which gives $\beta_j = D(\varphi) \circ \text{pr}_{s(\varphi)} \circ \alpha$. Now, noting that $s(\varphi) = i$, and that $\beta_i = \text{pr}_i \circ \alpha$, we finally get $\beta_j = D(\varphi) \circ \beta_i$. Hence, $(L(D), (\beta_i)_{i \in I})$ is a cone for D .

Let us now show that it is a limit cone. Let X be an object of $\mathcal{C}$, and $\gamma_i: X \rightarrow D(i)$ be maps such that $(X, (\gamma_i)_{i \in I})$ is a cone on D . Then, the collection of γ_i also forms a cone on the *discrete* diagram $I \rightarrow \mathcal{C}$, sending i to $D(i)$, hence factors through a unique map $\delta: X \rightarrow \prod_{i \in I} D(i)$. Now, consider the two compositions

$$\begin{array}{ccc}
 X & \xrightarrow{\delta} & \prod_{k \in I} D(k) \\
 & & \downarrow D \circ \text{pr}_s \\
 & & \prod_{\psi \in M} D(t(\psi))
 \end{array}
 \quad
 \begin{array}{ccc}
 & \xrightarrow{\text{pr}_t} & \\
 & \searrow & \\
 & & \prod_{\psi \in M} D(t(\psi))
 \end{array}$$

To show that they are equal, it is sufficient to check that for any map $\varphi: i \rightarrow j$ in $\mathcal{I}$, the postcomposition with pr_φ yields equal results. But, by the previous discussion, post-composing pr_t with pr_φ is equal to pr_j , and postcomposing $D \circ \text{pr}_s$ with pr_φ yields $D(\varphi) \circ \text{pr}_i$. Thus, we need to check that $\text{pr}_j \circ \delta = D(\varphi) \circ \text{pr}_i \circ \delta$ for all $\varphi: i \rightarrow j$ in $\mathcal{I}$. On the other hand, by assumption, $(X, (\gamma_i)_{i \in I})$ is a cone on D , and thus $\gamma_j = D(\varphi) \circ \gamma_i$. The result then follows since, for all $i \in I$, $\gamma_i = \text{pr}_i \circ \delta$, by definition of δ . Now, the universal property of the equalizer gives the existence of a unique map

$$h: X \rightarrow L(D)$$

Such that $\delta = \alpha \circ h$. This also implies that $\gamma_i = \text{pr}_i \circ \delta = \text{pr}_i \circ \alpha \circ h = \beta_i \circ h$ for all $i \in I$, and thus, h also factors the cone $(X, (\gamma_i)_{i \in I})$. We need to show

that h does so uniquely (observe that the unicity condition we already have on h is not quite in the desired form). Assume that we are given some other map $h': X \rightarrow L(D)$ such that for any $i \in I$, $\gamma_i = h' \circ \beta_i$. Then, for any $i \in I$, $\text{pr}_i \circ \delta = \text{pr}_i \circ \alpha \circ h'$. This implies that $\delta = \alpha \circ h'$, by the universal property of the product. Hence, $h' = h$, by the previous unicity. Finally, we have shown that $(L(D), (\beta_i)_{i \in I})$ is a limit cone for D . $\square$

Remark 4.3.10. In practice, we will almost always use this theorem to prove that a category is complete. This will amount to showing by hand that products and either equalizers or pullbacks exist. Usually by exhibiting a formula for those, and then deducing that the category is complete. When it comes to actually computing a limit, then depending on context, we may either use the explicit construction of $L(D)$ which appeared in the proof of Theorem 4.3.9, or "guess" the limit, and show by hand that it satisfies the universal property.

Corollary 4.3.11. *The category $\underline{\text{Set}}$ is complete*

Proof. By Theorem 4.3.9, it is enough to show that arbitrary products and equalizers exist in $\underline{\text{Set}}$, but this was done in Examples 4.3.3 and 4.3.5 respectively. $\square$

Let us finish this (sub)-section with the following unintuitive result.

Proposition 4.3.12. *Let $\mathcal{C}$ be a category. Assume that*

- $\mathcal{C}$ is complete
- $\mathcal{C}$ is small

Then, for any two objects X and Y in $\mathcal{C}$, $\text{Hom}_{\mathcal{C}}(X, Y)$ is either empty or a singleton. i.e. $\mathcal{C}$ must be a pre-ordered set, seen as a category.

Proof. Since $\mathcal{C}$ is assumed to be small, its set of morphism $\text{mor}(\mathcal{C})$ must have some cardinality λ . Now, assume that there exists two distinct parallel maps in $\mathcal{C}$, $f \neq g: X \rightarrow Y$, and consider the product $\prod_{k \in \lambda} Y$, i.e. the limit of the constant functor from λ seen as a discrete category to $\mathcal{C}$, sending every element $k \in \lambda$ to Y . It must exist, since $\mathcal{C}$ is assumed to be complete. The universal property of the limit tells us that maps from X to this product correspond exactly to arbitrary choices of maps $\alpha_k: X \rightarrow Y$ for $k \in \lambda$. We can produce such maps by choosing for each $k \in \lambda$ either f or g . This gives

$$2^\lambda \simeq \mathcal{P}(\lambda) \hookrightarrow \text{Hom}_{\mathcal{C}}(X, \prod_{\lambda} Y) \subset \text{mor}(\mathcal{C}) \simeq \lambda$$

Thus, passing to cardinality gives the inequality $2^\lambda \leq \lambda$, which is a contradiction (since λ must be at least 2, since we have at least two distinct maps f and g). Thus, there cannot exist two parallel distinct maps in $\mathcal{C}$. $\square$

Remark 4.3.13. The above proposition is surprisingly easy to prove for a statement with such consequences. It means that for all intents and purposes, the category we are interested in cannot be assumed to be both small and complete.

It is somewhat common in a lot of references about category theory to freely assume that any category is small (see the discussion in section 1.5). One sees with the above result that this cannot be done carelessly without running into contradictions. In fact, if one really wants to get rid of the smallness hypothesis, it is possible to do so (and indeed, most of the above mentioned references are in fact rigorous in their approach to category theory, or at least can be made so in a straightforward way). And one may actually want to do so, seeing as this hypothesis keeps appearing throughout the text. See Alternative Definition 1.2.3 (which cannot be phrased for general categories), Definition 2.2.6, or even the lack, so far, of a Yoneda Lemma (it will appear in Chapter 5), whose power is far lesser with this kind of restriction.

Working with this point of view, however forces one to be much more careful when considering limits (and categories of diagrams in general). Indeed, in those context where we want to work with the assumption that "every category is small, in some sense", we need to always be mindful of the cardinality of all categories involved. The notion of completeness, without further qualification, becomes somewhat meaningless, and is replaced by the notion of λ -completeness, where λ can be any cardinal. A category is λ -complete, if for any category $\mathcal{I}$ where the cardinality of $\text{mor}(\mathcal{I})$ is less than or equal to λ , and any diagram $D: \mathcal{I} \rightarrow \mathcal{C}$, the limit of D exists in $\mathcal{C}$. On the other hand, we may say that $\mathcal{C}$ is λ -small if the cardinality of $\text{mor}(\mathcal{C})$ is smaller than or equal to λ (recall that every category is small, so we can speak of the set of morphisms in that case).

In this context, the above proposition then becomes "for any cardinal λ , if $\mathcal{C}$ is a category that is both λ -small and λ -complete, then $\mathcal{C}$ is a preorder". In particular, we still retain the intuition, "a category cannot be both small and complete" it has to be in a sense, less small than it is complete, i.e. if it is λ -small and κ -complete, then we must have $\lambda > \kappa$. The point of view taken in those notes was chosen specifically to avoid those constant subtleties without sacrificing any of the rigor. The price to pay was a slightly less powerful theory.

4.3.3 Limits and adjunctions

The computation of limits and of right adjoints happen to be closely related in two ways. First, in suitably nice cases (for example when a category is complete), there is actually a functor associating its limit to any diagram, which can be seen to be a right adjoint. This is the content of Proposition 4.3.14. On the other hand, right adjoint in generals can be shown to **preserve limits** in a very precise sense. This is the content of Proposition 4.3.15.

Proposition 4.3.14. *Let $\mathcal{C}$ be a category, and $\mathcal{I}$ be a small category. Then, $\mathcal{C}$ admits all $\mathcal{I}$ -shaped limits if and only if the functor $\Delta: \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$ admits a right adjoint.*

In that case, the right adjoint $\lim_{\leftarrow}: \mathcal{C}^{\mathcal{I}} \rightarrow \mathcal{C}$ sends a diagram D to its limit $\lim_{\leftarrow} D$.

Proof. Let us first assume that there exists a functor $G: \mathcal{C}^{\mathcal{I}} \rightarrow \mathcal{C}$, such that $\Delta \dashv G$ is an adjoint pair. Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram (hence an object of $\mathcal{C}^{\mathcal{I}}$), and let us show that $G(D)$ is a limit for D . First, we need to exhibit the limit cone. Observe that the co-unit of the adjunction gives a map $\epsilon_D: \Delta(G(D)) \rightarrow D$, which is precisely a cone on D . Let us prove that this cone is terminal in $\text{Cone}(D)$.

Let $\alpha: \Delta X \rightarrow D$, where X is an object of $\mathcal{C}$ be an arbitrary cone on D . The adjunction $\Delta \dashv G$ gives us a map $\theta^{-1}(\alpha): X \rightarrow G(D)$, applying Δ to this map, then composing with ϵ_D gives the composition

$$\Delta X \xrightarrow{\Delta(\theta^{-1}(\alpha))} \Delta G(D) \xrightarrow{\epsilon_D} D$$

α

Where, by definition of θ , the two horizontal maps compose to α (see the proof of Proposition 3.1.4). Hence, we have proven that any cone on D must factor through ϵ_D . It remains to be proven that this factorization is unique. let $f: X \rightarrow G(D)$ be a map in $\mathcal{C}$ such that the following diagram is commutative.

$$\begin{array}{ccc} \Delta X & \xrightarrow{\Delta f} & \Delta G(D) \\ & \searrow \alpha & \downarrow \epsilon_D \\ & & D \end{array}$$

Apply G to the entire diagram, and precompose with η_X to produce the following commutative diagram (composed of the solid black arrows)

$$\begin{array}{ccccc} & & G(D) & & \\ & \nearrow f & & \nwarrow \eta_{G(D)} & \\ X & \xrightarrow{\eta_X} & G(\Delta X) & \xrightarrow{G(\Delta f)} & G(\Delta G(D)) \\ & & \searrow G(\alpha) & & \downarrow G(\epsilon_D) \\ & & & & G(D) \end{array}$$

$\theta^{-1}(\alpha)$

Now, the composition $G(\alpha) \circ \eta_X$ is equal to $\theta^{-1}(\alpha)$ (see Proposition 3.1.4), hence we can add the dotted arrow to the commutative diagram. Furthermore, since η is a natural transformation, adding the two blue arrows still yields a commutative diagram. Finally, observe that the composition $G(\epsilon_D) \circ \eta_{G(D)}$ is equal to Id_D , as one of the assumption on an adjunction, hence we must have

$\theta^{-1}(\alpha) = f$, which proves the unicity of the factorization. Finally, we have proven that the limit of D exists, and that $\lim D \simeq G(D)$.

For the other direction, we either need some (strong) version of the axiom of choice, or strengthen the hypothesis so that a *particular choice* of limit cone $(G(D), \alpha_D)$ has been made for each diagram D (note that for the direction we just proved, we produced a choice of limit cone for each diagram D through $G(D)$ and ϵ_D). In either case, we will be able to define $G: \mathcal{C}^{\mathcal{I}} \rightarrow \mathcal{C}$ on objects via $G(D) = \lim_{\leftarrow} D$, for some choice of representative. Now, given a map in $\mathcal{C}^{\mathcal{I}}$, $f: D \rightarrow D'$, we need to define a map in $\mathcal{C}$ $G(f): G(D) \rightarrow G(D')$. But, since we know that $G(D)$ and $G(D')$ are limits, through the universal property of limits, it is sufficient to define a map $\Delta G(D) \rightarrow D'$ in $\mathcal{C}^{\mathcal{I}}$. We proceed as follows. First, consider the limit cone $\alpha: \Delta G(D) \rightarrow D$. For all objects i in $\mathcal{I}$, set $\beta_i = f_i \circ \alpha_i: G(D) \rightarrow D'(i)$. We need to show that $G(D)$ together with the β_i , for $i \in \text{obj}(\mathcal{I})$ forms a cone on D' . Let $\varphi: i \rightarrow j$ be a map in $\mathcal{I}$, and consider the following diagram

$$\begin{array}{ccccc}
 & & D(i) & \xrightarrow{f_i} & D'(i) \\
 & \nearrow \alpha_i & \downarrow D(\varphi) & & \downarrow D'(\varphi) \\
 G(D) & & & & \\
 & \searrow \alpha_j & \downarrow & & \downarrow \\
 & & D(j) & \xrightarrow{f_j} & D'(j)
 \end{array}$$

The left triangle is commutative, since the α_i form a cone, and the square is commutative, since $f: D \rightarrow D'$ is a natural transformation. Furthermore, the top and bottom compositions are equal to β_i and β_j respectively. Hence, we have the equality $D'(\varphi) \circ \beta_i = \beta_j$, which proves that $G(D)$ together with the β_i , for $i \in \text{obj}(\mathcal{I})$ form a cone on D' . The universal property of the limit then gives a unique factorization of this cone through some map $G(f): G(D) \rightarrow G(D')$. This defines the action of G on morphisms.

Exercise 51. Use the universal property of limits to prove that G is indeed a functor, i.e.

- if $f: D \rightarrow D'$ and $g: D' \rightarrow D''$ are two maps in $\mathcal{C}^{\mathcal{I}}$, prove that $G(g \circ f) = G(g) \circ G(f)$
- $G(\text{Id}_D) = \text{Id}_{G(D)}$.

We now need to prove that G is a right adjoint to Δ . Let X be an object of $\mathcal{C}$, and D an object on $\mathcal{C}^{\mathcal{I}}$. Consider the following composition

$$\begin{array}{ccc}
 \text{Hom}_{\mathcal{C}}(X, G(D)) & \xrightarrow{\Delta} & \text{Hom}_{\mathcal{C}^{\mathcal{I}}}(\Delta X, \Delta G(D)) \\
 & \searrow & \downarrow \text{Hom}_{\mathcal{C}^{\mathcal{I}}}(\Delta X, \alpha_D) \\
 & & \text{Hom}_{\mathcal{C}^{\mathcal{I}}}(\Delta X, D)
 \end{array} \tag{4.12}$$

Since by hypothesis, $(G(D), \alpha_D)$ is a limit cone for D , the diagonal map must be a bijection. Now, what remains to be proven is that it is natural. This is the content of the following (non-trivial) exercise.

Exercise 52. • Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor, show that the map, defined for all X, Y objects of $\mathcal{C}$

$$F: \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is natural in X and Y .

- Prove that the horizontal maps in (4.12) is natural in X and D (hint : you may want to look back at the results at the end of Section 2.2).
- Prove that the collection of $\alpha_D: \Delta G(D) \rightarrow D$ for D objects of $\mathcal{C}^{\mathcal{I}}$ forms a natural transformation.
- Deduce that the vertical map in (4.12) is natural in X and D and conclude.

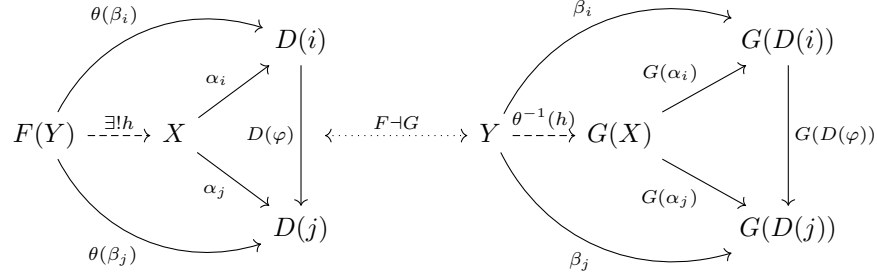
□

Proposition 4.3.15. *Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. If there exists $F: \mathcal{D} \rightarrow \mathcal{C}$ such that $F \dashv G$ is an adjoint pair, then for any diagram $D: \mathcal{I} \rightarrow \mathcal{C}$, and any limit cone of D , $(X, (\alpha_i)_{i \in \text{obj}(\mathcal{I})})$, $(G(X), (G(\alpha_i))_{i \in \text{obj}(\mathcal{I})})$ is a limit cone for the diagram $G \circ D: \mathcal{I} \rightarrow \mathcal{D}$. We will say in that case that G **preserves limits**.*

Proof. Let F be a left adjoint to G , and $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram, together with a limit cone on D , formed by an object of $\mathcal{C}$, X , and maps $\alpha_i: X \rightarrow D(i)$, for $i \in \text{obj}(\mathcal{I})$. Let us first prove that $G(X)$, and the maps $G(\alpha_i): G(X) \rightarrow G(D(i))$ form a cone on $G \circ D$. Let $\varphi: i \rightarrow j$ be a map in $\mathcal{I}$. By assumption, we have $\alpha_j = D(\varphi) \circ \alpha_i$. Applying G then gives $G(\alpha_j) = G(D(\varphi)) \circ G(\alpha_i)$, which proves that the $G(\alpha_i)$ form a cone on $G \circ D$.

Let us now prove that it is a limit cone. Let Y be an object of $\mathcal{D}$, and $\beta_i: Y \rightarrow G(D(i))$, for $i \in \text{obj}(\mathcal{I})$ be a family of maps forming a cone on $G \circ D$. In particular, for any map in $\mathcal{I}$, $\varphi: i \rightarrow j$, we must have $\beta_j = G(D(\varphi)) \circ \beta_i$. On the other hand, the adjunction $F \dashv G$ gives a family of maps $\theta(\beta_i): F(Y) \rightarrow D(i)$, for any $i \in \text{obj}(\mathcal{I})$. Furthermore, for any map $\varphi: i \rightarrow j$ in $\mathcal{I}$, by Lemma 3.4.1, we have the equality $\theta(\beta_j) = D(\varphi) \circ \theta(\beta_i)$, hence $(F(Y), (\theta(\beta_i))_{i \in \text{obj}(\mathcal{I})})$ is a cone on D . The universal property of the limit then gives the existence of a unique map $h: F(Y) \rightarrow X$ such that $\theta(\beta_i) = \alpha_i \circ h$ for all $i \in \text{obj}(\mathcal{I})$. Using Lemma 3.4.1 once more, we have that $\theta^{-1}(h): Y \rightarrow G(X)$ factors the cone $(Y, (\beta_i))$ through

$(G(X), (G(\alpha_i)))$. The situation is summarized in the following diagram



We just need to show that $\theta^{-1}(h): Y \rightarrow G(X)$ is the unique map such that $\beta_i = G(\alpha_i) \circ \theta^{-1}(h)$ for all $i \in \text{obj}(\mathcal{I})$. Let $g: Y \rightarrow G(X)$ be such that $G(\alpha_i) \circ g = \beta_i$ for all $i \in \text{obj}(\mathcal{I})$. Then, by Lemma 3.4.1, we must have $\theta(\beta_i) = \alpha_i \circ \theta(g)$, for all $i \in \text{obj}(\mathcal{I})$. The unicity of h then implies that $\theta(g) = h$, but since θ is a bijection, we finally have $g = \theta^{-1}(h)$. This proves the claim that $(G(X), (G(\alpha_i))_{i \in \text{obj}(\mathcal{I})})$ is a limit cone for $G \circ D$. $\square$

Remark 4.3.16. There are some partial reciprocal for Proposition 4.3.15. See for example [Mac71, Theorem 2, chapter V.6, p120]. Note however that the categories for which this theorem applies fall in the case detailed by proposition 4.3.12. Even though they are limited, the existence of those reciprocals is still useful as a first test when one suspect that a functor may be a right adjoint : does it preserve limits ? (this is especially relevant if the source category happens to be complete). If the answer is yes, then chances are, the functor is a right adjoint and it becomes a matter of exhibiting the left adjoint, and if the answer is no, then we know it is not.

4.4 Colimits

As mentionned earlier, every concept we have seen so far in this chapter can be dualized. Dualizing the notion of limits gives the notion of colimits. In this section, we make this process explicit. Throughout, we will refer to the particular Definitions and properties being dualized.

4.4.1 Definition by dualization

Before defining limits, we need to define the category of cocones.

Definition 4.4.1 (Dual of Definition 4.1.8). *Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram. Define the category of cocones on D , $\text{coCone}(D)$ to be the category with*

- *Objects are cocones $(X, \alpha: D \rightarrow \Delta X)$, where $\alpha: D \rightarrow \Delta X$ is a natural transformation.*

- Given two cocones (X, α) and (Y, β) , morphisms are the morphisms $f: X \rightarrow Y$ in $\mathcal{C}$ such that the following diagram is commutative in $\mathcal{C}^{\mathcal{I}}$

$$\begin{array}{ccc} & D & \\ \alpha \swarrow & & \searrow \beta \\ \Delta X & \xrightarrow{\Delta f} & \Delta Y \end{array}$$

In particular, $\text{Hom}_{\text{coCone}(D)}((X, \alpha), (Y, \beta))$ is defined as a subset of $\text{Hom}_{\mathcal{C}}(X, Y)$

- Identity and compositions are defined by restricting those of $\mathcal{C}$.

Exercise 53. Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram. Write D^{op} for the corresponding functor $D^{\text{op}}: \mathcal{I}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$. Show that, with the above definition

$$\text{coCone}(D) = (\text{Cone}(D^{\text{op}}))^{\text{op}}$$

(Observe that, for $(X, \alpha^{\text{op}}), (Y, \beta^{\text{op}})$ two cones in $\text{Cone}(D^{\text{op}})$, $\text{Hom}_{\text{Cone}(D^{\text{op}})}((X, \alpha^{\text{op}}), (Y, \beta^{\text{op}})) \subset \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y)$.)

Definition 4.4.2 (Defining the colimit as dual to the limit). *Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram, and $\alpha: D \rightarrow \Delta X$, be a cocone. It is a **colimit cocone** (sometimes abusively called **colimit cone** or, even worse, **limit cocone**) if $\alpha^{\text{op}}: \Delta X \rightarrow D^{\text{op}}$ is a limit cone in $\text{Cone}(D^{\text{op}})$.*

Remark 4.4.3. The above definition is almost never relevant in practical computations. However, it allows one to deduce all of the properties of the colimit from those of the limit. See Section 4.4.4.

4.4.2 Universal property of colimits

Let us rephrase the definition of colimits in more familiar terms.

Alternative Definition 4.4.4 (Dual of Definition 4.2.5). *Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram. The colimit of D , if it exists, is the initial object of $\text{coCone}(D)$.*

Once more, we can rephrase this definition in terms of universal properties.

Alternative Definition 4.4.5 (Dual of Alternative Definition 4.2.8). *Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a diagram. A **colimit cocone** is the data of*

- some object X in $\mathcal{C}$,
- for all $i \in \text{obj}(\mathcal{I})$, maps $\alpha_i: D(i) \rightarrow X$

Such that, for any map $\varphi: i \rightarrow j$ in $\mathcal{I}$, the following diagram commutes

$$\begin{array}{ccc} D(i) & & \\ \downarrow D(\varphi) & \searrow \alpha_i & \\ & & X \\ & \nearrow \alpha_j & \\ D(j) & & \end{array}$$

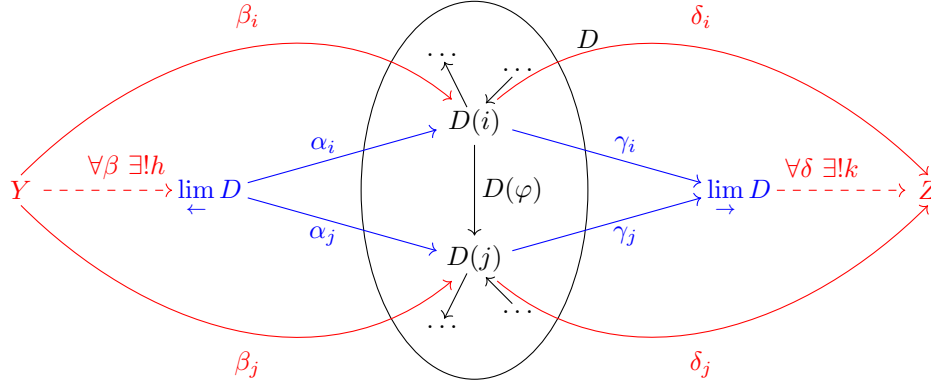


Figure 4.2: The universal property of limit cones and colimit cocones

And such that for any other cone $(Y, (\beta_i)_{i \in \text{obj}(\mathcal{I})})$, satisfying the same condition, there exists a unique map $h: X \rightarrow Y$ such that for all $i \in \text{obj}(\mathcal{I})$, $\beta_i = h \circ \alpha_i$.

Remark 4.4.6. We can now complete Figure 4.1 with the colimit cocone. See Figure 4.2. On the left, we have in blue the limit cone, and in red an arbitrary cone. On the right, in blue the colimit cocone, and in red, an arbitrary cocone. We will denote $\varprojlim D$ for the limit, and $\varinjlim D$ for the colimit. This figure should serve as a motivation for this notation. Given a diagram $\varprojlim D$ and $\varinjlim D$ should be thought of as sitting immediately "before" and "after" the diagram, when thinking in terms of the flow of arrows. For the same reasons, in old texts, the limit, $\varprojlim D$ is sometimes called the **inverse limit**, because one has to go against the flow of arrows (from the diagram) to reach the limit, whereas the colimit, $\varinjlim D$, is called the **direct limit**, because it is reached by following the flow of arrows from the diagram. Note that any given diagram may admit both a limit cone and a colimit cocone, only one of them, or neither.

Exercise 54. Convince yourself that Definition 4.4.2 and Alternative Definitions 4.4.4 and 4.4.5 are equivalent.

Exercise 55. Let $\mathcal{I}$ be the discrete category with two objects.

- Let $D: \mathcal{I} \rightarrow \underline{\text{Set}}$ be the functor with values X and Y , two sets. Show that $\varinjlim D$ is the disjoint union $X \amalg Y$ (Hint : dualize everything in Motivation 1.1.1).
- Let $D: \mathcal{I} \rightarrow \mathbf{k}\text{-Vect}$. Show that the objects $\varprojlim D$ and $\varinjlim D$ are isomorphic (Hint : compute them first).

4.4.3 Computations in $\underline{\text{Set}}$

Let us look at a few examples of colimits in $\underline{\text{Set}}$

Example 4.4.7 (The empty colimit). Consider $\mathcal{I} = \emptyset$ the empty category with no object and no morphisms, and the unique functor $D: \mathcal{I} \rightarrow \underline{\text{Set}}$. Its colimit must be an initial object of $\text{coCone}(D)$, which can be seen to be equivalent to $\underline{\text{Set}}$ itself (Dualize Example 4.1.9). $\underline{\text{Set}}$ admits for an initial object the empty set, $\emptyset$. Indeed, given any set X , there exists a unique map $\emptyset \rightarrow X$, the empty map.

Example 4.4.8 (Pairwise disjoint unions). Let $\mathcal{I}$ be the discrete category with two objects. We have seen in Exercise 55 that given a diagram $D: \mathcal{I} \rightarrow \underline{\text{Set}}$, its colimit is the disjoint union.

Example 4.4.9 (Arbitrary disjoint unions). Let $\mathcal{I}$ be a discrete category, with the set $I = \text{obj}(\mathcal{I})$ of arbitrary cardinality. And let $D: \mathcal{I} \rightarrow \underline{\text{Set}}$ be a diagram. Then, the colimit of D is the disjoint union :

$$\lim_{\rightarrow} D \simeq \coprod_{i \in I} D(i)$$

The colimit cocone is formed by considering the inclusions $\iota_j: D(j) \hookrightarrow \coprod_{i \in I} D(i)$

Exercise 56. Show that the cocone formed by the inclusions ι_j , for $j \in I$ is indeed a colimit cocone. (Hint : you can dualize the reasoning of Example 4.3.3, and check that the arguments also hold).

Example 4.4.10 (Co-equalizers). Let $\mathcal{I}$ be the category with two parallel maps. i.e. $\mathcal{I}$ has two objects, 1 and 2, and two parallel maps from 1 to 2, $\varphi: 1 \rightarrow 2$ and $\psi: 1 \rightarrow 2$, in addition to the two identity maps, Id_1 and Id_2 . A diagram $D: \mathcal{I} \rightarrow \underline{\text{Set}}$ is the data of

- two sets, $X = D(1)$ and $Y = D(2)$,
- two parallel maps, $f = D(\varphi): X \rightarrow Y$ and $g = D(\psi): X \rightarrow Y$

Following Example 4.3.5, we will find that the data of a cocone reduces to the data of a set Z , together with a map $\alpha: Y \rightarrow Z$, such that $\alpha \circ f = \alpha \circ g$. The colimit cocone of D will be obtained by considering a specific quotient of Y . Consider on Y the equivalence relation, denoted by $\sim$, generated by

$$f(x) \sim g(x), \forall x \in X$$

i.e. the relation $\sim$ is the smallest transitive, reflexive and symmetric relation containing the pairs $(f(x), g(x))$ for all $x \in X$.

Exercise 57. Let $y \neq y' \in Y$, show that $y \sim y'$ if and only if there exists $n \geq 1$, and a sequence of elements in X , $(x_i)_{1 \leq i \leq n}$ such that

- $y = f(x_1)$ or $y = g(x_1)$
- for all $1 \leq i \leq n-1$, $f(x_i) = g(x_{i+1})$ or $g(x_i) = f(x_{i+1})$
- $y' = f(x_n)$ or $y' = g(x_n)$.

Let $Z = Y/\sim$ be the quotient under the equivalence relation $\sim$, and $p: Y \rightarrow Z$ be the quotient map. We will show that (Z, p) is the colimit cocone. Let us first see that it is a cone. Let $x \in X$, and consider the two compositions $p \circ f(x)$ and $p \circ g(x)$. By definitions, they are equal to equivalence classes, $[f(x)]$ and $[g(x)]$ respectively. By definition of the equivalence relation, we have $[f(x)] = [g(x)]$, hence $p \circ f = p \circ g$, which proves that (Z, p) is a cocone.

Now, let $\alpha: Y \rightarrow W$ be any map such that $\alpha \circ f = \alpha \circ g$. Let us show that it must factor through some map $h: Z \rightarrow W$. We define $h: Z \rightarrow W$ as follows. Any element in Z is an equivalence class $[y]$ for some $y \in Y$, and we can define $h([y]) = \alpha(y)$. We need to check that h is well defined. Let y and y' be two elements in Y such that $y \sim y'$. If they are equal, then there is nothing to check. Otherwise, there must exist some sequence $x_1, \dots, x_n$, as in Exercise 57. But now, by assumption, $\alpha \circ f(x_i) = \alpha \circ g(x_i)$, for all $1 \leq i \leq n$, hence, $\alpha(y) = \alpha(y')$. This means that the definition of $h([y])$ does not depend on a choice of representative for the equivalence class $[y]$. In particular, h is well-defined, and satisfies $\alpha = h \circ p$. In addition, h is necessarily unique since the condition $\alpha = h \circ p$ determines h on each equivalence class. This proves that (Z, p) is the colimit cocone. We will call it the **co-equalizer** of f and g , sometimes denoted $\text{coEq}(f, g)$.

Example 4.4.11 (Pushouts). Let $\mathcal{I}$ be the category with three objects, 0, 1 and 2, and two non-identity morphisms, $\varphi: 0 \rightarrow 1$ and $\psi: 0 \rightarrow 2$. A diagram $D: \mathcal{I} \rightarrow \underline{\text{Set}}$ then correspond to the following data:

$$\begin{array}{ccc} X = D(0) & \xrightarrow{f=D(\varphi)} & Y = D(1) \\ g=D(\psi) \downarrow & & \\ Z = D(2) & & \end{array}$$

As we have seen several time before, specifying a cocone on D amounts to choosing a set, W , and two maps $\alpha_Y: Y \rightarrow W$ and $\alpha_Z: Z \rightarrow W$, such that $\alpha_Y \circ f = \alpha_Z \circ g$. Now, a reasoning dual to Example 4.3.6 gives that the colimit cocone should be a quotient of $Y \amalg Z$, under some equivalence relation, $\sim$, generated by, for $y \in Y$ and $z \in Z$

$$\exists x \in X, f(x) = y \text{ and } g(x) = z \Rightarrow y \sim z$$

Exercise 58. Prove that the quotient of $Y \amalg Z$ by the above equivalence relation is indeed a colimit (Hint : dualize the arguments of Example 4.3.6).

This particular shape of colimit is called a **pushout** (or push-out). Given the colimit cocone

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow \iota_Y \\ Z & \xrightarrow{\iota_Z} & W \end{array}$$

We may say that " ι_Y is the pushout of g along f ", and we will often write $Y \cup_X Z$ or $Y \amalg_X Z$ to stand for the colimit W . We may even write, to be more specific,

$Y \cup_g Z$. Furthermore, whenever a square of this form is a colimit cocone, we will simply state that it is a **pushout square**, or that W is the pushout, which we may simply write $W = Y \cup_X Z$.

Exercise 59. We aim to compute some specific pushouts.

- Let E be a set, and $Y, Z \subset E$ be two subsets, Show that the following square is both a pullback square and a pushout square.

$$\begin{array}{ccc} Y \cap Z & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & Y \cup Z \end{array}$$

- Let $A \subset X$ be the inclusion of a non-empty subset. Show that the square

$$\begin{array}{ccc} A & \longrightarrow & \{*\} \\ \downarrow & & \downarrow i \\ X & \xrightarrow{p} & X/A \end{array}$$

is a pushout square. Where, X/A is the quotient of X under the equivalence relation $\sim$, where, for $x, x' \in X$, we have $x \sim x'$ if and only if $x = x'$ or x and x' are both elements of A . In this case, $p: X \rightarrow X/A$ is the quotient map, and i sends $*$ to the equivalence class of elements of A .

4.4.4 Properties of colimits

All abstracts properties of colimits can be deduced from those of limits by dualizing. Nevertheless, it may be useful to have those states explicitly, so we recall them here.

Exercise 60. Write the proof for each of those statements by dualizing the arguments of the previous sections.

The dual notion to completeness is **cocompleteness**, we briefly state its definition below.

Definition 4.4.12 (Dual of Definition 4.3.7). *Let $\mathcal{C}$ be a category, and $\mathcal{I}$ some fixed small category. We say that $\mathcal{C}$ **admits all $\mathcal{I}$ -shaped colimits** if, for all diagram $D: \mathcal{I} \rightarrow \mathcal{C}$, the colimit $\lim_{\rightarrow} D$ exists. When $\mathcal{I}$ is a discrete category with two objects, we will say that $\mathcal{C}$ admits all pairwise coproduct. If $\mathcal{I}$ is the category with two parallel maps, we will say that $\mathcal{C}$ admits all co-equalizers.*

If $\mathcal{C}$ admits all $\mathcal{I}$ shaped colimits for $\mathcal{I}$ any small discrete category, we will say that $\mathcal{C}$ admits all coproducts.

*If $\mathcal{C}$ admits all $\mathcal{I}$ -shaped colimits for $\mathcal{I}$ any small category, we will say that $\mathcal{C}$ is **cocomplete**.*

*If a category is both complete and cocomplete, we will say that it is **bicomplete**.*

Theorem 4.4.13 (Dual of Theorem 4.3.9). *Let $\mathcal{C}$ be a category. The following are equivalent.*

- $\mathcal{C}$ is cocomplete,
- $\mathcal{C}$ admits arbitrary coproducts and coequalizers,
- $\mathcal{C}$ admits arbitrary products and pullbacks.

Corollary 4.4.14. *The category $\mathbf{Set}$ is bicomplete.*

Proof. By Corollary 4.3.11 is complete. By Theorem 4.4.13, it is enough to show that $\mathbf{Set}$ admits arbitrary coproducts and coequalizers to deduce that it is cocomplete, but this was done in Examples 4.4.9 and 4.4.10. $\square$

Proposition 4.4.15 (Dual of Proposition 4.3.12). *Let $\mathcal{C}$ be a category. Assume that*

- $\mathcal{C}$ is cocomplete,
- $\mathcal{C}$ is small.

Then, for any two objects X and Y in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is either empty or a singleton. i.e. $\mathcal{C}$ must be a pre-ordered set, seen as a category.

Proposition 4.4.16 (Dual of Proposition 4.3.14). *Let $\mathcal{C}$ be a category, and $\mathcal{I}$ a small category. Then, $\mathcal{C}$ admits all $\mathcal{I}$ shaped colimits if and only if the functor $\Delta: \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$ admits a left adjoint.*

In that case, the left adjoint $\lim_{\rightarrow}: \mathcal{C}^{\mathcal{I}} \rightarrow \mathcal{C}$ sends a diagram D to its colimit $\lim_{\rightarrow} D$.

Proposition 4.4.17 (Dual of Proposition 4.3.15). *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. If there exists $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $F \dashv G$ is an adjoint pair, then for any diagram $D: \mathcal{I} \rightarrow \mathcal{C}$, and any colimit cocone of D , $(X, (\alpha_i)_{i \in \mathrm{obj}(\mathcal{I})})$, $(F(X), F(\alpha_i)_{i \in \mathrm{obj}(\mathcal{I})})$ is a colimit cocone for the diagram $F \circ D: \mathcal{I} \rightarrow \mathcal{D}$. We will say that F **preserves colimits**.*

4.5 Examples of computations of limits and colimits

4.5.1 Elementary examples

Let us consider a few examples of computations of limits and colimits. It may be a good exercise to redo the computations by yourself when reading through the examples.

Example 4.5.1. Let $\mathcal{I}$ be the category obtained by considering the ordered set $\mathbb{N}$ as a category. That is, objects are integers $0, 1, 2, \dots$, and $\text{Hom}_{\mathcal{I}}(k, n)$ is a singleton, if $k \leq n$ or empty if $k > n$. Consider the diagram $D: \mathcal{I} \rightarrow \underline{\mathbb{R} - \text{Vect}}$ sending n to $\mathbb{R}^n$, and, for all $k < n$, sending the map $k \rightarrow n$ to the map

$$\begin{aligned} i_{k,n}: \mathbb{R}^k &\rightarrow \mathbb{R}^n \\ (x_1, \dots, x_k) &\mapsto (x_1, \dots, x_k, 0, \dots, 0) \end{aligned}$$

We aim to compute the colimit of D . We will show that it is given by the vector space $\mathbb{R}^\infty$, defined, as a set, via

$$\mathbb{R}^\infty = \{(x_n)_{n \in \mathbb{N}} \mid \forall n \in \mathbb{N}, x_n \in \mathbb{R}, \exists N \in \mathbb{N}, \forall k \geq N, x_k = 0\}$$

Addition and scalar multiplication are then defined termwise. The cocone colimit will be given by the maps

$$\begin{aligned} \iota_n: \mathbb{R}^n &\rightarrow \mathbb{R}^\infty \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_n, 0, \dots) \end{aligned}$$

Note that by definition, $\iota_k = \iota_n \circ i_{k,n}$, for all $k < n$, hence it is a cocone on D . Let us prove that it is indeed a colimit cocone. Let V be a vector space, and $\alpha_n: \mathbb{R}^n \rightarrow V$ be a family of maps forming a cocone. We must have, for all $k < n \in \mathbb{N}$, $\alpha_k = \alpha_n \circ i_{k,n}$. Now, we define $h: \mathbb{R}^\infty \rightarrow V$ as follows. Given an element of $\mathbb{R}^\infty$, $x = (x_n)_{n \in \mathbb{N}}$, there must be some $N \in \mathbb{N}$ such that $x_n = 0$ for all $n \geq N$. But then, $x = \iota_N(x_1, \dots, x_N)$, by definition of ι_N . We can define $h(x) = \alpha_N(x_1, \dots, x_N)$.

Now, assume that there exists $K < N$ such that we also have $x = \iota_K(x_1, \dots, x_K)$. Then, $x = \iota_N \circ i_{K,N}(x_1, \dots, x_K) = \iota_N(x_1, \dots, x_K, 0, \dots, 0)$. In particular, we must have $x_n = 0$ for all $n > K$. Thus, we have $h(x) = \alpha_N(x_1, \dots, x_N) = \alpha_N(i_{K,N}(x_1, \dots, x_K)) = \alpha_K(x_1, \dots, x_K)$. Hence, the definition of h did not depend on a choice for N , and is thus unambiguous. It is also clear by construction that $\alpha_n = h \circ \iota_n$, for all $n \in \mathbb{N}$, thus h factors the cocone α . Furthermore, the hypothesis that $\alpha_n = h \circ \iota_n$, for all $n \in \mathbb{N}$ fully determines h on each of the elements of $\mathbb{R}^\infty$, thus it must be unique. This proves that $\mathbb{R}^\infty$ is the colimit of D . We may want to write

$$\mathbb{R}^\infty = \bigcup_{n \in \mathbb{N}} \mathbb{R}^n$$

and think of this particular colimit as an increasing union (by identifying $\mathbb{R}^k$ with the subset $i_{k,n}(\mathbb{R}^k) \subset \mathbb{R}^n$ whenever $k \leq n$, or more generally, identifying $\mathbb{R}^k$ with $\iota_k(\mathbb{R}^k) \subset \mathbb{R}^\infty$). We may also write

$$\lim_{\rightarrow} (\mathbb{R}^0 \hookrightarrow \mathbb{R}^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \dots) \simeq \mathbb{R}^\infty.$$

Example 4.5.2. Let $\mathcal{I}$ be the opposite of the category used in the previous example. That is, objects of $\mathcal{I}$ are still the integers $0, 1, 2, \dots$, but, given $k, n \in$

$\mathbb{N}$, $\text{Hom}_{\mathcal{I}}(k, n)$ is empty if $k < n$ and a singleton if $k \geq n$. Consider the diagram $D: \mathcal{I} \rightarrow \underline{\mathbb{R}} - \text{Vect}$ sending n to $\mathbb{R}^n$, and for $k \geq n$, sending $k \rightarrow n$ to the map

$$\begin{aligned} p_{k,n}: \mathbb{R}^k &\rightarrow \mathbb{R}^n \\ (x_1, \dots, x_k) &\mapsto (x_1, \dots, x_n) \end{aligned}$$

We aim to compute the limit of D . We will show that it is given by the vector space $\mathbb{R}^{\times\infty}$, defined as

$$\mathbb{R}^{\times\infty} = \{(x_n)_{n \in \mathbb{N}} \mid \forall n \in \mathbb{N}, x_n \in \mathbb{R}\}$$

where addition and multiplication with a scalar are defined termwise. the limit cone is given by the maps

$$\begin{aligned} \pi_n: \mathbb{R}^{\times\infty} &\rightarrow \mathbb{R}^n \\ (x_k)_{k \in \mathbb{N}} &\mapsto (x_1, \dots, x_n) \end{aligned}$$

By construction, we have $\pi_n = p_{k,n} \circ \pi_k$, whenever $k \geq n$, hence it is indeed a cone. Let V be a vector space, and $\alpha_n: V \rightarrow \mathbb{R}^n$ be a family of maps forming a cone on D . We must have $\alpha_n = p_{k,n} \circ \alpha_k$ for all $k \geq n$. We define $h: V \rightarrow \mathbb{R}^{\times\infty}$ as follows. Let $v \in V$ be a vector, we want to define $h(v) = (y_n)_{n \in \mathbb{N}}$. We will do so term by term. Let $k \in \mathbb{N}$, then $\alpha_k(v) = (x_1, \dots, x_k) \in \mathbb{R}^k$. We set $y_k = x_k$. Doing so for each $k \in \mathbb{N}$ defines $h(v)$. Now, observe that since the α_n , for $n \in \mathbb{N}$, form a cone, whenever $n \geq k$, we must have $p_{n,k} \circ \alpha_k = \alpha_n$, hence we have $\alpha_n(v) = (y_1, \dots, y_n) = \pi_n(h(v))$, and the cone $(V, (\alpha_n)_{n \in \mathbb{N}})$ indeed factors through h . But h must be unique since the equality $\pi_n \circ h = \alpha_n$ determines the first n terms of $h(v)$ for all $v \in V$. Since this holds for all $n \in \mathbb{N}$, h is unique, and we have shown that $\mathbb{R}^{\times\infty}$ is the limit of D . We may summarize this as follows :

$$\mathbb{R}^{\times\infty} \simeq \varprojlim (\cdots \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R}^1 \rightarrow \mathbb{R}^0)$$

Exercise 61. Let $\mathcal{I}$ be the discrete category with set of objects $\mathbb{N}$. Consider the functor $D: \mathcal{I} \rightarrow \underline{\mathbb{R}} - \text{Vect}$ sending every object to $\mathbb{R}$. Show that

- $\varinjlim D \simeq \mathbb{R}^\infty$, from Example 4.5.1.
- $\mathbb{R}^{\times\infty} \simeq \varprojlim D$, from Example 4.5.2

For vector spaces, it is common to write the coproduct

$$\mathbb{R}^\infty \simeq \bigoplus_{n \in \mathbb{N}} \mathbb{R}$$

and the product

$$\mathbb{R}^{\times\infty} \simeq \prod_{n \in \mathbb{N}} \mathbb{R}$$

- Exhibit a map $\mathbb{R}^\infty \rightarrow \mathbb{R}^{\times\infty}$ in three ways

- by hand,
- using the universal property of the limit,
- using the universal property of the colimit.
- Show that those three maps coincide.
- Show that it is not an isomorphism.
- Compare with Exercise 55, what changed ?
- (Harder) Consider the subcategory $\mathcal{I}_n \subset \mathcal{I}$, containing only the integers $\leq n$, and let $D_n: \mathcal{I}_n \rightarrow \mathbb{R} - \mathbf{Vect}$ be the restriction of D .
 - Show that $\lim_{\rightarrow} D_n \simeq \mathbb{R}^n \simeq \lim_{\leftarrow} D_n$.
 - Express the diagrams appearing in Examples 4.5.1 and 4.5.2 in terms of the $\lim_{\rightarrow} D_n$ and $\lim_{\leftarrow} D_n$ respectively.
 - Try to interpret the first two questions with the above decomposition.

Example 4.5.3. Let $f: V \rightarrow W$ be a linear map. i.e. a morphism in the category of vector spaces $\mathbb{R} - \mathbf{Vect}$. We may always consider the morphism $0: V \rightarrow W$, sending every element of V to the zero vector in W . Both $\text{Eq}(f, 0)$, and $\text{coEq}(f, 0)$ are known objects. They are in fact the kernel and cokernel of f . We will prove both claims. First, observe that $\ker(f)$ is defined as a subobject of V

$$\ker(f) = \{v \in V \mid f(v) = 0_W\}$$

where 0_W is the neutral element of W . Hence, it comes with a map (the inclusion map) $i: \ker(f) \rightarrow V$. Let us show that this map is the equalizer of f and 0 . Let $\varphi: V' \rightarrow V$ be a map in $\mathbb{R} - \mathbf{Vect}$ such that $f \circ \varphi = 0 \circ \varphi = 0$, then, for all $v \in V'$, we have $f(\varphi(v)) = 0$, hence, $\varphi(v) \in \ker(f)$. This means that φ must factor through $\ker(f)$. Since $\ker(f) \rightarrow V$ is an inclusion, this factorization is necessarily unique, which proves that $\ker(f) \simeq \text{Eq}(f, 0)$.

For the cokernel, recall that the cokernel is defined as the quotient

$$\text{coker}(f) = W/\text{Im}(f)$$

where $\text{Im}(f) \subset W$ is the image of f . By construction, it comes with a projection map $p: W \rightarrow \text{coker}(f)$. Let us show that it is a coequalizer. First, clearly $p \circ f = 0 = p \circ 0$. Now, let $\psi: W \rightarrow W'$ be a map such that $\psi \circ f = 0$. We need to define $h: \text{coker}(f) \rightarrow W'$ factoring ψ . Any element in $\text{coker}(f)$ is an equivalence class $[w]$, for w a vector in W . Furthermore, if w and w' are two elements of W such that $[w] = [w']$, it means that $w' - w$ is in the image of f . In particular, there must be some $v \in V$ such that $f(v) = w' - w$. Hence, we must have $\psi(w') - \psi(w) = \psi(f(v)) = 0$, this means that $\psi(w') = \psi(w)$, and we can unambiguously define $h([w]) = \psi(w)$. This defines a map $h: \text{coker}(f) \rightarrow W'$, which factors ψ . Since this factorization is unique, we have shown that $\text{coker}(f) \simeq \text{coEq}(f, 0)$.

Example 4.5.4. Consider the category Pre-Ord of pre-ordered sets. Objects of Pre-Ord are pre-ordered sets, that is, they are sets, X , together with a reflexive and transitive relation, that we will usually denote $\leq$. Given two pre-ordered sets, $(X, \leq_X)$ and $(Y, \leq_Y)$ morphisms between them are order-preserving morphisms. i.e. they are set-morphisms $f: X \rightarrow Y$ such that for any $x_1, x_2 \in X$ we have

$$x_1 \leq_X x_2 \Rightarrow f(x_1) \leq_Y f(x_2)$$

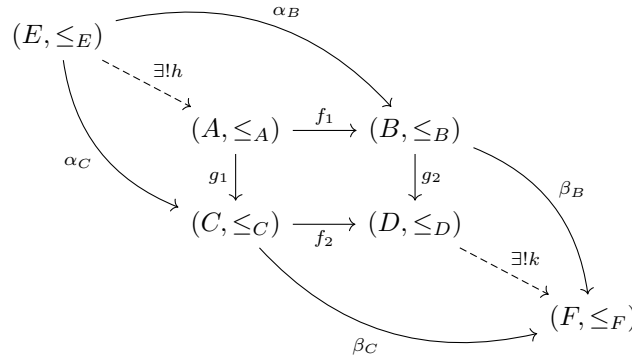
Compositions and identities are the same as for Set. Now, consider the following four pre-ordered sets

- $(A, \leq_A)$, where $A = \{a, b, c\}$, and $\leq_A$ is reduced to the identity.
- $(B, \leq_B)$, where $B = \{a, b, c\}$, and $a \leq_B b$, with no relations between c and either a or b ,
- $(C, \leq_C)$, where $C = \{a, b, c\}$, and $b \leq_C c$, with no relations between a and either b or c ,
- $(D, \leq_D)$, where $D = \{a, b, c\}$, and $a \leq_D b$, $b \leq_D c$ and $a \leq_D c$.

Consider the following diagram

$$\begin{array}{ccc} (A, \leq_A) & \xrightarrow{f_1} & (B, \leq_B) \\ g_1 \downarrow & & \downarrow g_2 \\ (C, \leq_C) & \xrightarrow{f_2} & (D, \leq_D) \end{array}$$

where each map is given by the identity of the underlying set $\{a, b, c\} \rightarrow \{a, b, c\}$. (Note that those are not identity maps in Pre-Ord, since the domains and codomains are not equal). The square is both a pullback square and a pushout square. To see this, consider a cone and a cocone, as in the following picture



Let us consider the cone first. We must have $g_2 \circ \alpha_B = f_2 \circ \alpha_C$. This means that, at the level of sets, the two maps $\alpha_B: E \rightarrow \{a, b, c\}$ and $\alpha_C: E \rightarrow \{a, b, c\}$ must be equal. Thus, we can define $h: E \rightarrow A = \{a, b, c\}$ via, $h = \alpha_B = \alpha_C$, it

indeed factors the cone, and is necessarily unique. We just need to check that h is indeed a map in Pre-Ord. Let $x_1, x_2 \in E$ be two elements such that $x_1 \leq_E x_2$. We must have $\alpha_B(x_1) \leq_B \alpha_B(x_2)$, which means that either $\alpha_B(x_1) = \alpha_B(x_2)$ or $\alpha_B(x_1) = a$ and $\alpha_B(x_2) = b$ (this is the only non-trivial relation in $(B, \leq_B)$). On the other hand, we must have $\alpha_C(x_1) \leq_C \alpha_C(x_2)$ hence either $\alpha_C(x_1) = \alpha_C(x_2)$ or $\alpha_C(x_1) = b$ and $\alpha_C(x_2) = c$. Since $\alpha_B = \alpha_C$, the only way for both conditions to be satisfied simultaneously is that $\alpha_B(x_1) = \alpha_B(x_2)$, hence $h(x_1) = h(x_2)$, and h is indeed a map in Pre-Ord, which proves the pullback square claim.

For the cocone, similarly, we can define $k = \beta_B = \beta_C$ at the level of sets. Let us now prove that k is a map in Pre-Ord. There is a condition to check for every pair of comparable elements in D .

- for $a \leq_D b$, since β_B is a map in Pre-Ord and since $a \leq_B b$, we must have $k(a) = \beta_B(a) \leq_F \beta_B(b) = k(b)$.
- for $b \leq_D c$, similarly, we have $k(b) = \beta_C(b) \leq_F \beta_C(c) = k(c)$.
- for $a \leq_D c$, since by assumption $\leq_F$ is a transitive relation, we have $k(a) \leq_F k(b) \leq_F k(c)$ and hence $k(a) \leq_F k(c)$. In fact, this check was automatic from transitivity.

Finally, we have proven that the square is also a pushout square.

As a final remark, observe that this square is entirely trivial at the level of sets, yet there are meaningful things happening with the *structure* that those sets carry. We will study this phenomenon in detail in the next section.

4.5.2 Preorders, posets and a non-trivial computation

Before stating the goal of this section, let us start with some definitions. Recall from Example 4.5.4 the definition of the category Pre-Ord. Objects are pre-ordered sets, that is, sets equipped with a transitive and reflexive relation, and morphisms are order-preserving maps. We will be interested in the full subcategory Poset $\subset$ Pre-Ord of partially ordered sets, often called posets. Those are the objects $(P, \leq_P)$ such that, for all $p, q \in P$, we have

$$p \leq_P q \text{ and } q \leq_P p \Rightarrow p = q$$

By abuse of language, we will call the above condition anti-symmetry.

The goal of this section is to prove the following claim

Proposition 4.5.5. *The category Poset is bicomplete.*

and to provide a way of computing limits and colimits in Poset. The process should also serve as an illustration on how to approach this kind of question in any concrete category.

We will start with the easier half of the claim, specifically, that Poset is complete. We will do so using Proposition 4.3.15 by considering a forgetful functor Poset $\rightarrow$ Set, showing that it is a right adjoint, and thus must preserve

limits. And use this observation to "guess" what limits should be in $\mathbf{Poset}$. Start by considering the functor $\text{Forget}: \mathbf{Poset} \rightarrow \mathbf{Set}$ sending a poset $(P, \leq_P)$ to its underlying set P , and a map $f: (P, \leq_P) \rightarrow (Q, \leq_Q)$ to the underlying map $f: P \rightarrow Q$. This functor admits a left adjoint $\text{Disc}: \mathbf{Set} \rightarrow \mathbf{Poset}$ sending the set X , to the discrete poset $\text{Disc}(X)$ with underlying set X , and no relations between elements of X .

Lemma 4.5.6. *The pair $\text{Disc} \dashv \text{Forget}$ is an adjoint pair.*

Proof. Consider the tautological map

$$\begin{aligned} \text{Hom}_{\mathbf{Set}}(X, \text{Forget}(P, \leq_P)) &\rightarrow \text{Hom}_{\mathbf{Poset}}(\text{Disc}(X), (P, \leq_P)) \\ (f: X \rightarrow P) &\mapsto (f: \text{Disc}(X) \rightarrow (P, \leq_P)) \end{aligned}$$

To see that it is well defined, observe that maps in $\mathbf{Poset}$ from $\text{Disc}(X)$ to $(P, \leq_P)$ are set maps from X to P , with no extra conditions since there are no non-trivial relations in $\text{Disc}(X)$. This also shows that it is a bijection, since by definition $\text{Hom}_{\mathbf{Poset}}(\text{Disc}(X), (P, \leq_P)) \subset \text{Hom}_{\mathbf{Set}}(X, P)$ is defined as a subset, but the absence of extra conditions means that it is the entire set. Finally, naturality of this bijection follows from the fact that composition in $\mathbf{Poset}$ is defined by restriction of that of $\mathbf{Set}$. $\square$

Now, Lemma 4.5.6 together with Proposition 4.3.15 tell us that the functor $\text{Forget}: \mathbf{Poset} \rightarrow \mathbf{Set}$ must preserve limits. This is not enough to know that limits *exist*. But we know that *if they do*, their underlying sets must correspond to limits computed in $\mathbf{Set}$.

Let us make the above observations explicit. Let $D: \mathcal{I} \rightarrow \mathbf{Poset}$ be a diagram. Assuming that $\lim_{\leftarrow} D$ exists, we must have $\text{Forget}(\lim_{\leftarrow} D) \simeq \lim_{\leftarrow} (\text{Forget} \circ D)$. Since all limits exist in $\mathbf{Set}$ (Corollary 4.3.11), to show that $\lim_{\leftarrow} D$ exists, it is enough to find the correct order relation, $\leq$ such that $(\lim_{\leftarrow} (\text{Forget} \circ D), \leq)$ is a limit for D . Let us fix some notation. Let $X \simeq \lim_{\leftarrow} (\text{Forget} \circ D)$, and $\alpha_i: X \rightarrow \text{Forget}(D(i))$, $i \in \text{obj}(\mathcal{I})$ be a limit cone for $\text{Forget} \circ D$. We will write $\leq_{D(i)}$ for the order relation in the poset $D(i)$, for $i \in \mathcal{I}$. For $x, x' \in X$, define the order relation $\leq$ via

$$x \leq x' \Leftrightarrow \alpha_i(x) \leq_{D(i)} \alpha_i(x'), \forall i \in \text{obj}(\mathcal{I}) \quad (4.13)$$

We must check that $\leq$ is transitive reflexive and anti-symmetric, but the first two property will be inherited from each of the $\leq_{D(i)}$. For anti-symmetry, assume that x and x' are such that $x \leq x'$ and $x' \leq x$. Then for each $i \in \text{obj}(\mathcal{I})$, we have $\alpha_i(x) \leq_{D(i)} \alpha_i(x')$ and $\alpha_i(x') \leq_{D(i)} \alpha_i(x)$. Anti-symmetry of $\leq_{D(i)}$ then implies that $\alpha_i(x) = \alpha_i(x')$. But this implies that $x = x'$. Indeed, consider the singleton $Y = \{*\}$ and the maps $\beta_i: Y \rightarrow \text{Forget}(D(i))$ mapping $*$ to $\alpha_i(x)$ for all $i \in \text{obj}(\mathcal{I})$. This must form a cone, since the maps β_i can be obtained by restricting the α_i to the subset $\{x\} \subset X$, and the commutativity properties of the α_i are preserved under restriction. Since X is assumed to be the limit, this

cone must factor uniquely through some map $Y \rightarrow X$. If x and x' were distinct elements, we would get two distinct factorizations, one mapping $*$ to x , the other mapping $*$ to x' , hence, $x = x'$, and we have proven anti-symmetry.

Now, let us prove that $(X, \leq)$ is indeed the limit. First, we need to check that $((X, \leq), (\alpha_i)_{i \in \text{obj}(\mathcal{I})})$ is still a cone. The maps $\alpha_i: (X, \leq) \rightarrow D(i)$ are order preserving maps, by definition, since $\leq$ was constructed such that $x \leq x' \Rightarrow \alpha_i(x) \leq_{D(i)} \alpha_i(x')$. Furthermore, for any map in $\mathcal{I}$, $\varphi: i \rightarrow j$, we still have the equality $D(\varphi) \circ \alpha_i = \alpha_j$ since it just needs to be checked at the level of sets. To see that it is a limit cone, let $(P, \leq_P)$ be a poset, and $\beta_i: (P, \leq_P) \rightarrow D(i)$, for $i \in \mathcal{I}$ be a collection of maps forming a cone. Since X is the limit in Set, we must have a unique map $h: P \rightarrow X$ factoring the β_i in Set. The only thing to check is that $h: (P, \leq_P) \rightarrow (X, \leq)$ is a map in Poset. Let $p, q \in P$ be such that $p \leq_P q$. Then, by assumption $\beta_i(p) \leq_{D(i)} \beta_i(q)$ for all $i \in \text{obj}(\mathcal{I})$. Since $\beta_i = \alpha_i \circ h$ for all $i \in \text{obj}(\mathcal{I})$, we must have $\alpha_i(h(p)) \leq_{D(i)} \alpha_i(h(q))$, for all $i \in \text{obj}(\mathcal{I})$. By definition of $\leq$, this means that $h(p) \leq h(q)$, and we have proven that $h: (P, \leq_P) \rightarrow (X, \leq)$ is a map in Poset. Which finally proves that $(X, \leq)$ is the limit of D in Poset.

We have proved the first half of Proposition 4.5.5. Now, to see why the second half will be more difficult, consider the following example.

Example 4.5.7. Consider the following three posets

- $(P, \leq_P)$, with $P = \{a, b\}$ and no relations
- $(Q, \leq_Q)$, with $Q = \{a, b\}$ and the relation $a \leq_Q b$,
- $(R, \leq_R)$, with $R = \{a, b\}$ and the relation $b \leq_R a$.

We want to compute the following pushout

$$\begin{array}{ccc} (\{a, b\}, \leq_P) & \xrightarrow{f} & (\{a, b\}, \leq_Q) \\ g \downarrow & & \downarrow \alpha_Q \\ (\{a, b\}, \leq_R) & \xrightarrow{\alpha_R} & ? \end{array}$$

where f and g are simply the identity at the level of sets. Based on what we did so far, we may expect the pushout to be the set $\{a, b\}$ together with some relation $\leq$. However, observe that the condition that α_Q and α_R are maps in Poset give contradictory constraints. Indeed, since $a \leq_Q b$, we must have $\alpha_Q(a) \leq \alpha_Q(b)$. On the other hand, we must have $\alpha_R(b) \leq \alpha_R(a)$. But since the square needs to be commutative in the end, we end up with the incompatible assumption: $a \leq b$ and $b \leq a$. This is only possible if $a = b$, and indeed the correct pushout square is the following

$$\begin{array}{ccc} (\{a, b\}, \leq_P) & \xrightarrow{f} & (\{a, b\}, \leq_Q) \\ g \downarrow & & \downarrow \alpha_Q \\ (\{a, b\}, \leq_R) & \xrightarrow{\alpha_R} & \{*\} \end{array}$$

where $\{*\}$ stands for the unique poset (up to unique isomorphism) with only one element. To see this, consider an arbitrary cocone, $\beta_Q: (Q, \leq_Q) \rightarrow (S, \leq_S)$ and $\beta_R: (R, \leq_R) \rightarrow (S, \leq_S)$. Since it is a cocone, we must have $\beta_Q(a) = \beta_R(a) = x$ and $\beta_Q(b) = \beta_R(b) = y$. Furthermore, as we have just seen, the condition on maps in Poset will force the relations $x \leq_S y$ and $y \leq_S x$. Hence, since $\leq_S$ has to be anti-symmetric, we will have $x = y$. But then, the cocone factors uniquely through the map $\{*\} \rightarrow S$ sending $*$ to x .

The above example shows that we will have to be more subtle to prove that Poset is cocomplete. To prove it, we need to temporarily return to the category of pre-orders Pre-Ord. In fact, we will show the following.

Proposition 4.5.8. *The category Pre-Ord is bicomplete.*

To prove it, consider the functor $\text{Forget}' : \text{Pre-Ord} \rightarrow \text{Set}$, sending a pre-order $(A, \leq_A)$ to A and a map $f : (A, \leq_A) \rightarrow (B, \leq_B)$ to $f : A \rightarrow B$. It admits a left adjoint $\text{Disc}' : \text{Set} \rightarrow \text{Pre-Ord}$, similar to the functor Disc that we have seen before. All proofs, up to completeness, are essentially the same as the one we just did for Poset, so we will not repeat them.

On the other hand Forget' also admits a right adjoint, $\text{Total} : \text{Set} \rightarrow \text{Pre-Ord}$. It sends a set X to the pre-order with underlying set X , and with the relation $x \leq_{\text{Total}} x'$ for all $x, x' \in X$.

Lemma 4.5.9. *The pair $\text{Forget}' \dashv \text{Total}$ is an adjoint pair.*

Proof. Consider the tautological map

$$\begin{aligned} \text{Hom}_{\text{Pre-Ord}}((A, \leq_A), \text{Total}(X)) &\rightarrow \text{Hom}_{\text{Set}}(\text{Forget}'(A, \leq_A), X) \\ (f : (A, \leq_A) \rightarrow \text{Total}(X)) &\mapsto (f : A \rightarrow X) \end{aligned}$$

The set $\text{Hom}_{\text{Pre-Ord}}((A, \leq_A), \text{Total}(X))$ is defined as the subset of $\text{Hom}_{\text{Set}}(A, X)$ containing the order-preserving map and the above map is the inclusion. Since the order relation on $\text{Total}(X)$ is such that all pairs are comparable in both direction, any map is an order-preserving map. Hence, the above map is a bijection. Naturality follows from the fact that composition in Pre-Ord is inherited from Set. $\square$

Now, Lemma 4.5.9, together with Proposition 4.4.17 implies that the functor $\text{Forget}' : \text{Pre-Ord} \rightarrow \text{Set}$ preserves colimits (in addition to preserving limits). Now, let $D : \mathcal{I} \rightarrow \text{Pre-Ord}$ be a diagram. Let $X \simeq \varinjlim (\text{Forget}' \circ D)$ be the colimit in Set, with the cocone given by maps $\alpha_i : \text{Forget}'(D(i)) \rightarrow X$, for $i \in \text{obj}(\mathcal{I})$. We will write $\leq_i$ for the pre-order relation in the pre-order $D(i)$. Let $\leq$ be the transitive relation on X generated by

$$c \leq_i d \Rightarrow \alpha_i(c) \leq \alpha_i(d). \quad (4.14)$$

That is, $x \leq y$ in X if and only if, there exists a sequence of elements of X , $(x_1, \dots, x_n)$ such that, $x = x_1$, $y = x_n$ and for each $1 \leq k \leq n-1$, there

exists some $i \in \text{obj}(\mathcal{I})$ and some $c, d \in D(i)$ such that $c \leq_i d$ and $\alpha_i(c) = x_k$, $\alpha_i(d) = x_{k+1}$. This relation will be reflexive, since any $x \in X$ must be in the image of some α_i , for some $i \in \text{obj}(\mathcal{I})$.

Let us show that $(X, \leq)$ is a colimit cocone for D . First, observe that for all $i \in \text{obj}(\mathcal{I})$, the maps $\alpha_i: D_i \rightarrow (X, \leq)$ are maps in Pre-Ord, since they are order-preserving by construction, hence we indeed have a cocone. Let $(A, \leq_A)$ be a pre-order, and for $i \in \text{obj}(\mathcal{I})$, $\beta_i: D(i) \rightarrow (A, \leq_A)$ be maps forming a cocone. Then, since X is the colimit in set, there exists a unique factorization $h: X \rightarrow A$ such that for all $i \in \text{obj}(\mathcal{I})$, $\beta_i = h \circ \alpha_i$. Now, the only thing to show is that $h: (X, \leq) \rightarrow (A, \leq_A)$ is a map in Pre-Ord. Let $x, y \in X$ be two elements such that $x \leq y$. Then there must be some sequence in X , $(x_1, \dots, x_n)$ such that $x_1 = x$, $x_n = y$, and for any $1 \leq k \leq n-1$, there exists some i_k and some elements $c_k, d_k \in D(i_k)$ such that $c_k \leq_i d_k$, $\alpha_{i_k}(c_k) = x_k$ and $\alpha_{i_k}(d_k) = x_{k+1}$. But now, for $1 \leq k \leq n-1$, since β_{i_k} is an order-preserving map, we have

$$\begin{aligned} h(x_k) &= h(\alpha_{i_k}(c_k)) \\ &= \beta_{i_k}(c_k) \\ &\leq_A \beta_{i_k}(d_k) \\ &= h(\alpha_{i_k}(d_k)) \\ &= h(x_{k+1}) \end{aligned}$$

transitivity of $\leq_A$ then implies that $h(x) \leq_A h(y)$, which proves that h is order preserving. Hence, that $(X, \leq)$ is the colimit of D in Pre-Ord.

We now go back to the category of posets, Poset. Let $\text{Incl}: \text{Poset} \hookrightarrow \text{Pre-Ord}$ be the inclusion of the subcategory. We now define a functor $\text{Quotient}: \text{Pre-Ord} \rightarrow \text{Poset}$ as follows. Given a pre-order $(A, \leq_A)$, let $\sim_A$ be the equivalence relation defined by, for $a, b \in A$,

$$a \leq_A b \text{ and } b \leq_A a \Leftrightarrow a \sim_A b.$$

Observe that the transitivity of $\leq_A$ guarantees that $\sim_A$ is also transitive. Now, consider on $A/\sim_A$ the order relation, $\preceq_A$ defined by

$$[a] \preceq_A [b] \Leftrightarrow a \leq_A b$$

Note that transitivity of $\leq_A$ makes the relation $\preceq_A$ well defined. Indeed, if a, a' and b, b' are such that $a \leq_A b$, $[a] = [a']$, and $[b] = [b']$, then $a' \leq_A a \leq_A b \leq_A b'$, hence $a' \leq_A b'$, and there is no ambiguity in the above definition.

Now, if $f: (A, \leq_A) \rightarrow (B, \leq_B)$ is a map in Pre-Ord, observe that the following map is well-defined and order-preserving, by construction.

$$\begin{aligned} [f]: (A/\sim_A, \preceq_A) &\rightarrow (B/\sim_B, \preceq_B) \\ [a] &\mapsto [f(a)] \end{aligned}$$

Finally, observe that $(A/\sim_A, \preceq_A)$ is a poset. It is a pre-order, by construction, and let $[a], [b] \in A/\sim_A$ be two elements such that $[a] \preceq [b]$ and $[b] \preceq [a]$. Then, by construction, $a \leq_A b$ and $b \leq_A a$, hence $[a] = [b]$.

We have just defined a functor $\text{Quotient}: \underline{\text{Pre-Ord}} \rightarrow \underline{\text{Poset}}$ sending a pre-order $(A, \leq_A)$ to the poset $(A/\sim_A, \preceq_A)$, and sending a map $f: (A, \leq_A) \rightarrow (B, \leq_B)$ to the map $[f]$.

Lemma 4.5.10. *Let $(P, \leq_P)$ be a poset. Then*

$$\text{Quotient}(\text{Incl}(P, \leq_P)) \simeq (P, \leq_P).$$

More specifically, the composition $\text{Quotient} \circ \text{Incl}: \underline{\text{Poset}} \rightarrow \underline{\text{Poset}}$ is isomorphic to the identity of $\underline{\text{Poset}}$.

Proof. Consider the equivalence relation $\sim_P$. Its equivalence classes are singletons, thus $P/\sim_P$ is in canonical bijection with P . $\square$

Lemma 4.5.11. *The pair $\text{Quotient} \dashv \text{Incl}$ is an adjoint pair.*

Proof. Consider the following map

$$\begin{aligned} \theta: \text{Hom}_{\underline{\text{Pre-Ord}}}((A, \leq_A), \text{Incl}(P, \leq_P)) &\rightarrow \text{Hom}_{\underline{\text{Poset}}}(\text{Quotient}(A, \leq_A), (P, \leq_P)) \\ (f: A \rightarrow P) &\mapsto ([f]: A/\sim_A \rightarrow P/\sim_P \simeq P) \end{aligned}$$

Let us prove that it is a bijection. Let $g: \text{Quotient}(A, \leq_A) \rightarrow (P, \leq_P)$ be an order-preserving map. Define $f: A \rightarrow P$ via $f(a) = g([a])$ for all $a \in A$. This is an order-preserving map, since, if $a \leq_A b$, then $[a] \preceq_A [b]$ which implies that $f(a) = g([a]) \leq_P g([b]) = f(b)$. And, by construction, $\theta(f) = g$, hence θ is surjective. To see that it is injective, let $f, f': (A, \leq_A) \rightarrow (P, \leq_P)$ be two maps in $\underline{\text{Pre-Ord}}$ such that $\theta(f) = \theta(f')$. This means that, for any $a \in A$, $f(a) = \theta(f)([a]) = \theta(f')([a]) = f'(a)$, hence that $f = f'$. We have shown that θ is a bijection.

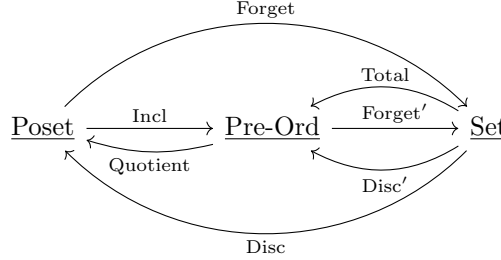
Exercise 62. Show that the map θ is natural in $(A, \leq_A)$ and $(P, \leq_P)$. $\square$

Now, we will conclude by applying Proposition 4.4.17 to the left adjoint $\text{Quotient}: \underline{\text{Pre-Ord}} \rightarrow \underline{\text{Poset}}$. Consider a diagram $D: \mathcal{I} \rightarrow \underline{\text{Poset}}$. Then, the composition $\text{Incl} \circ D: \mathcal{I} \rightarrow \underline{\text{Pre-Ord}}$ is a diagram in $\underline{\text{Pre-Ord}}$, and since $\underline{\text{Pre-Ord}}$ is cocomplete, it must admit a colimit, $(X, \leq)$ in $\underline{\text{Pre-Ord}}$, together with its cocone $\alpha_i: \text{Incl}(D(i)) \rightarrow (X, \leq)$. Now, since Quotient preserves colimits, $\text{Quotient}(X, \leq)$ is a colimit of $(\text{Quotient} \circ \text{Incl}) \circ D$. But, since $\text{Quotient} \circ \text{Incl}$ is isomorphic to the identity of $\underline{\text{Poset}}$ by Lemma 4.5.10, $\text{Quotient}(X, \leq)$ must in fact be the colimit of D . The cocone colimit is then given by the $[\alpha_i]: D(i) \rightarrow \text{Quotient}(X, \leq)$. We can summarize this computation by the formula

$$\lim_{\rightarrow} D \simeq \text{Quotient}(\lim_{\rightarrow} (\text{Incl} \circ D))$$

Finally, we have proven that the colimit $\lim_{\rightarrow} D$ exists, hence that $\underline{\text{Poset}}$ is cocomplete, which completes the proof of Proposition 4.5.5.

The complete line of reasoning is summarized in the following diagram, featuring every functor that appeared throughout.



We have the adjunctions $\text{Quotient} \dashv \text{Incl}$, $\text{Disc}' \dashv \text{Forget}'$, $\text{Forget}' \dashv \text{Total}$, and $\text{Disc} \dashv \text{Forget}$, that were all used at some point together with the Propositions 4.3.15 and 4.4.17 which state that right adjoints preserve limits and left adjoint preserve colimits. Furthermore, we used the isomorphism $\text{Quotient} \circ \text{Incl} \simeq \text{Id}_{\text{Poset}}$. Note the extra (unused) isomorphism, $\text{Forget}' \circ \text{Incl} \simeq \text{Forget}$ and $\text{Quotient} \circ \text{Disc}' \simeq \text{Disc}$.

Now, the reasoning went as follows. Since $\text{Forget}: \text{Poset} \rightarrow \text{Set}$ is a right adjoint, it preserves limits, hence computing limits in Poset amounts to computing it in Set first, then finding the correct order relation. For which we have formula (4.13).

For colimits, the situation is slightly more complicated. Given a diagram in Poset , we can use the inclusion functor, $\text{Incl}: \text{Poset} \rightarrow \text{Pre-Ord}$ to interpret it as a diagram in Pre-Ord . Then, since $\text{Forget}': \text{Pre-Ord} \rightarrow \text{Set}$ is a left adjoint, it preserve colimits. Hence, computing colimits in Pre-Ord amounts to computing it in Set first, then finding the correct relation, given by formula (4.14). Once the colimit is computed in Pre-Ord , apply the functor $\text{Quotient}: \text{Pre-Ord} \rightarrow \text{Poset}$ to recover the desired colimit in Poset . Observe that this is exactly the reasoning that we (implicitly) applied in Example 4.5.7.

Chapter 5

Categories of presheaves

In this chapter, we introduce categories of presheaves (sometimes abusively called presheaf categories). They are a very general class of categories which admits very good properties. They are ubiquitous in category theory : in logic, we have the theory of toposes studying systematically categories of presheaves (and subcategories of those), in topology and geometry, the category of simplicial sets and variants of it, have been proven to be a good combinatorial substitute for categories of spaces, and even category theory itself can be "embedded" into the theory of categories of presheaves. Furthermore, a lot of already interesting categories happen to be categories of presheaves (the category of directed graphs for example, see Example 5.1.5). We will start with the definition and a few simple examples, before explaining why categories of presheaves are special.

5.1 The basics of presheaves

5.1.1 Definition

Definition 5.1.1. *Let A be a small category. The category of presheaves on A , denoted $\hat{A}$ is simply the category*

$$\hat{A} = \text{Fun}(A^{\text{op}}, \underline{\text{Set}})$$

*Its objects are functors $X : A^{\text{op}} \rightarrow \underline{\text{Set}}$, they will be called **presheaves** (singular: **presheaf**). Morphisms, $f : X \rightarrow Y$ correspond to natural transformations. i.e. they consist of the data of maps $f_a : X(a) \rightarrow Y(a)$, for all $a \in \text{obj}(A)$, such that for any map $\varphi : a \rightarrow b$ in A , the following square commutes.*

$$\begin{array}{ccc} X(a) & \xrightarrow{f_a} & Y(a) \\ \uparrow X(\varphi) & & Y(\varphi) \uparrow \\ X(b) & \xrightarrow{f_b} & Y(b) \end{array}$$

Identities and compositions are defined object-wise. i.e. $(\text{Id}_X)_a = \text{Id}_{X(a)}: X(a) \rightarrow X(a)$, for all $a \in \text{obj}(A)$, and similarly for the composition.

Remark 5.1.2. Once again, we have redefined the category of functors. Observe that, by definition, we have the equality

$$\widehat{A} = \text{Fun}(A^{\text{op}}, \underline{\text{Set}}) = \underline{\text{Set}}^{A^{\text{op}}}$$

Just as in Remark 4.1.3, this is a matter of point of view. We don't want to *think* of an object of the category $\widehat{A}$ as a functor, even though *it is one*. Similarly, we don't want to think of an object X of $\widehat{A}$ as a diagram, because calling it a diagram suggest we may be interested in its limit and colimit, but in most cases $\varprojlim X$ and $\varinjlim X$, while perfectly valid mathematical objects, will be meaningless. Instead, we will mainly think of an object X of $\widehat{A}$ as collection of sets, the $X(a)$, indexed by objects a of A , with some **structure maps**, the maps $X(\varphi): X(b) \rightarrow X(a)$, for all maps $\varphi: a \rightarrow b$. Of course, this is just a rephrasing of the definition of a functor.

5.1.2 Examples

Let us start with two tautological examples.

Example 5.1.3. Let A be the empty category. Then $\widehat{A}$ contains a single object, the empty functor $A \rightarrow \underline{\text{Set}}$. Hence, $\widehat{A}$ is the category with a single object and only the identity map.

Example 5.1.4. Now, let A be the category with one object, and only the identity morphisms. Then, the data of a functor $A^{\text{op}} \rightarrow \underline{\text{Set}}$ is exactly the data of a set. Furthermore, natural transformations between such functors are exactly maps of sets. Hence, $\widehat{A} \simeq \underline{\text{Set}}$.

Now let us consider a large family of examples coming from graph theory.

Example 5.1.5. Let A be the category with two objects e and v , and two non-identity morphisms, $s: v \rightarrow e$ and $t: v \rightarrow e$. Let us prove that $\widehat{A}$ is equivalent to the category of directed graphs, $\underline{\text{Graph}}$.

Recall that a directed graph, G , is the data of a quadruple $(E(G), V(G), s_G, t_G)$, where $E(G)$ is a set of edges, $V(G)$ is a set of vertices, and the two maps $s_G, t_G: E(G) \rightarrow V(G)$ correspond to source and targets of edges. Maps of directed graphs, $f: G \rightarrow H$ are pairs of maps $f_E: E(G) \rightarrow E(H)$, and $f_V: V(G) \rightarrow V(H)$, such that $f_V \circ s_G = s_H \circ f_E$ and $f_V \circ t_G = t_H \circ f_E$. Now, we can define the functors

$$\begin{aligned} \widehat{A} &\rightarrow \underline{\text{Graph}} \\ X &\mapsto (X(e), X(v), X(s): X(e) \rightarrow X(v), X(t): X(e) \rightarrow X(v)) \\ f: X &\rightarrow Y \mapsto (f_e, f_v) \end{aligned}$$

Observe that the condition that $f: X \rightarrow Y$ is a natural transformation is equivalent to the condition that (f_e, f_v) is a map of graphs. In the other direction,

we have

$$\begin{array}{c} \text{Graph} \rightarrow \hat{A} \\ G \mapsto \left\{ \begin{array}{ll} A^{\text{op}} & \rightarrow \underline{\text{Set}} \\ v & \mapsto V(G) \\ e & \mapsto E(G) \\ s & \mapsto s_G \\ t & \mapsto t_G \end{array} \right. \end{array}$$

It sends a map of graphs $(f_E, f_V): G \rightarrow H$ to the natural transformation g with values $g_e = f_E$, $g_v = f_V$. It is clear, by construction, that this is an equivalence of categories. In fact, we already think of graphs as presheaves (even though we may not be aware of it).

Example 5.1.6. We can refine the previous example to define graphs with extra structure. For example, consider the category of directed graphs with edges equipped with a label from some set Σ . Those can be seen as tuples $(E(G), V(G), s_G, t_G, l_G: E(G) \rightarrow \Sigma)$. Maps between such labelled graphs will then be those maps that preserve the labelling, i.e. maps of graphs $f: G \rightarrow H$ such that, $l_H \circ f_E = l_G$. Let lab-Graph stand for the category of labelled graphs.

We may show that this is also equivalent to a category of presheaves. Consider the small category A , with an object for each element $\sigma \in \Sigma$, plus an extra object v . Non-identity maps are $s_\sigma: v \rightarrow \sigma$ and $t_\sigma: v \rightarrow \sigma$, for all $\sigma \in \Sigma$. Now, we define a functor

$$\begin{array}{c} \hat{A} \rightarrow \underline{\text{lab-Graph}} \\ X \mapsto (\coprod_{\sigma \in \Sigma} X(\sigma), X(v), \coprod_{\sigma \in \Sigma} X(s_\sigma), \coprod_{\sigma \in \Sigma} X(t_\sigma), l_X) \\ f: X \rightarrow Y \mapsto (\coprod_{\sigma \in \Sigma} f_\sigma, f_v) \end{array}$$

Where l_X is defined as, given an edge $\alpha \in X(\tau) \subset \coprod_{\sigma \in \Sigma} X(\sigma)$, $l_X(\alpha) = \tau$.

Its inverse is defined as follows

$$\begin{array}{c} \underline{\text{lab-Graph}} \rightarrow \hat{A} \\ (G, l_G) \mapsto \left\{ \begin{array}{ll} \sigma & \mapsto (l_G)^{-1}(\sigma) \subset E(G) \\ v & \mapsto V(G) \\ s_\sigma & \mapsto (s_G)|_{(l_G)^{-1}(\sigma)} \\ t_\sigma & \mapsto (t_G)|_{(l_G)^{-1}(\sigma)} \end{array} \right. \\ f: (G, l_G) \rightarrow (H, l_H) \mapsto ((f_\sigma)_{\sigma \in \Sigma}, f_v) \end{array}$$

Where f_σ is the restriction $(f_E)|_{(l_H)^{-1}(\sigma)}$.

Exercise 63. • Check that the two functors above are well-defined, and that they form an equivalence of categories.

- Consider the graph L with a single vertex, and the set of edges $E(L) = \Sigma$. Consider the overcategory $\mathbf{Graph}/L$, whose objects are pairs (G, f) , where G is a graph, and $f: G \rightarrow L$ is a map of graphs, and whose morphisms $\varphi: (G, f) \rightarrow (H, g)$ are graph morphisms $\varphi: G \rightarrow H$ such that the following diagram commutes.

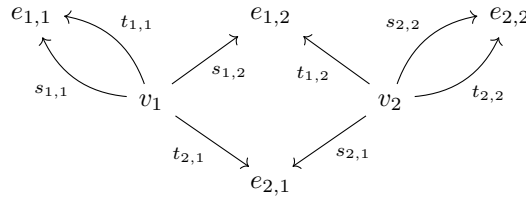
$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ & \searrow f \quad \swarrow g & \\ & L & \end{array}$$

Show that there exists an equivalence of categories $\mathbf{lab-Graph} \simeq \mathbf{Graph}/L$.

Example 5.1.7. Let $n \geq 2$ and consider the category of n -colored graphs, $n\text{-Col-Graph}$. Its objects are tuples $(E(G), V(G), s_G, t_G, c_G: V(G) \rightarrow \{1, \dots, n\})$, that is, they are graph together with a map from the set of vertices to a set of n colors. Morphisms $f: (G, c_G) \rightarrow (H, c_H)$ are morphisms of graphs $(f_E, f_V): G \rightarrow H$, such that $c_H \circ f_V = c_G$, i.e. such that f preserves the coloring of vertices. Compositions and identities are defined as those of $\mathbf{Graph}$.

We want to show that $n\text{-Col-Graph}$ is equivalent to a category of presheaves. This will be another variation on the category from Example 5.1.5. Let us define the small category A . We need one object for each possible kind of vertices, i.e. $v_1, v_2, \dots, v_n$. Furthermore, we will need more than one kind of edges. Indeed, if we try to have a single kind of edge, e , then we will have maps $s_1: v_1 \rightarrow e$, $s_2: v_2 \rightarrow e, \dots, s_n: v_n \rightarrow e$. When passing to presheaves, this would give maps $X(e) \rightarrow X(v_1)$, and $X(e) \rightarrow X(v_2)$ up to $X(e) \rightarrow X(v_n)$. This would mean that any given edge must have one source for each color of vertices. Clearly, this is not the notion we are trying to capture. On the other hand, we can imagine that each edge is partially colored by the vertices on its ends. Thus, we have n^2 many kinds of edges, depending on the color near the source, and near the target. Distinguishing between each kind of edges now gives a suitable category.

A will have for objects v_i , for $1 \leq i \leq n$, and $e_{i,j}$ for $1 \leq i, j \leq n$. And non-identity morphisms $s_{i,j}: v_i \rightarrow e_{i,j}$, and $t_{i,j}: v_j \rightarrow e_{i,j}$ for all $(i, j) \in \{1, \dots, n\}^2$. To make this clearer, consider the graph of the category A when $n = 2$.



Exercise 64. • Prove that there exists an equivalence $n\text{-Col-Graph} \simeq \hat{A}$.

- We say that a coloring c_G of a graph G is proper, if the map c_G is such that, whenever $v, v' \in V(G)$ are two vertices connected by an edge, $c_G(v) \neq c_G(v')$. Express the property that a graph is properly colored through its description as a presheaf.

- Consider the graph C with set of vertices $V(C) = \{1, \dots, n\}$, and set of edges $E(C) = \{1, \dots, n\}^2$, where $s_C(i, j) = i$ and $t_C(i, j) = j$. Show that there exists an equivalence $n\text{-Col-Graph} \simeq \underline{\text{Graph}}/C$, where $\underline{\text{Graph}}/C$ is an overcategory, defined as in Exercise 63.
- Show that the sub-category of properly colored graphs is also equivalent to a category of presheaves, and that it is equivalent to an overcategory $\underline{\text{Graph}}/C'$, where $C' \subset C$ is a well-chosen subgraph.

5.2 Representable presheaves

5.2.1 The Yoneda embedding

Definition 5.2.1. Let A be a small category, and a an object of A . Write h_a for the contravariant functor from Example 2.1.9,

$$\begin{aligned} h_a: A^{\text{op}} &\rightarrow \underline{\text{Set}} \\ b &\mapsto \text{Hom}_A(b, a) \\ (\varphi: b \rightarrow c) &\mapsto \text{Hom}_A(\varphi, a): \text{Hom}_A(c, a) \rightarrow \text{Hom}_A(b, a) \end{aligned}$$

It is an object of $\widehat{A}$ and will be called the **presheaf represented by a** . We may also say that it is a **representable presheaf** if we do not want to refer to the object a itself.

Definition 5.2.2. Given a small category A , the **Yoneda embedding** is the functor $y: A \rightarrow \widehat{A}$ defined by

$$\begin{aligned} A &\rightarrow \widehat{A} \\ a &\mapsto h_a \\ \varphi: a \rightarrow b &\mapsto h_\varphi: h_a \rightarrow h_b \end{aligned}$$

Where h_φ is the natural transformation defined in Exercise 21.

The name Yoneda "embedding" in the following section.

Exercise 65. Let A be the category with two objects, e and v and two non-identity morphisms, $s, t: v \rightarrow e$. We have seen in exercise 5.1.5 that $\widehat{A}$ is the category of graphs. What are the representable graphs h_e and h_v ? Make explicit the functor $y: A \rightarrow \underline{\text{Graph}}$.

5.2.2 The Yoneda lemma

The goal of this subsection is to state and prove the Yoneda lemma. It is probably the most well-known result in category theory, and its proof has confused generations of students. It contains no tricks, and is in fact a linear application of the definitions we have seen so far. We will go through it in details, in the hope of alleviating some of the usual confusions around it.

Theorem 5.2.3 (Yoneda Lemma). *Let A be a small category, X be an object of $\widehat{A}$ and a an object of A . There exists a bijection*

$$\mathrm{Hom}_{\widehat{A}}(h_a, X) \simeq X(a)$$

which is natural in a and X . More precisely, the two functors

$$\begin{aligned} A \times \widehat{A} &\rightarrow \underline{\mathrm{Set}} \\ (a, X) &\mapsto X(a) \\ (a, X) &\mapsto \mathrm{Hom}_{\widehat{A}}(h_a, X) \end{aligned}$$

are related by a natural isomorphism.

Proof. We start by defining the isomorphism explicitly, for a in A and X in $\widehat{A}$.

$$\begin{aligned} \phi_{a,X} : \mathrm{Hom}_{\widehat{A}}(h_a, X) &\rightarrow X(a) \\ (f : h_a \rightarrow X) &\mapsto f_a(\mathrm{Id}_a) \end{aligned}$$

To see why the above definition makes sense, note that a map in $\widehat{A}$, $f : h_a \rightarrow X$, is in fact the data of a family of maps $f_b : h_a(b) \rightarrow X(b)$ for all objects b of A . In particular, we can consider $f_a : h_a(a) \rightarrow X(a)$. Now, $h_a(a) = \mathrm{Hom}_A(a, a)$ by definition, thus we can pick the element $\mathrm{Id}_a \in h_a(a)$, and its image under f_a will be some element of $X(a)$, meaning that the definition of $\phi_{a,X}$ makes sense. Now we need to prove the following things:

- That $\phi_{a,X}$ is a bijection for all a in A and X in $\widehat{A}$.
- That $\phi_{a,X}$ is natural in X .
- That $\phi_{a,X}$ is natural in a .

For the first part, we will explicitly define an inverse. For $x \in X(a)$, we want to define a map $\psi_{a,X}(x) : h_a \rightarrow X$. This amounts to defining maps $(\psi_{a,X}(x))_b : h_a(b) \rightarrow X(b)$ for each objects b of A . Recall that elements of $h_a(b)$ are maps $\varphi : b \rightarrow a$ in A . Since X is a presheaf on A , we can consider the map $X(\varphi) : X(a) \rightarrow X(b)$. This map sends x to an element of $X(b)$. This leads to the definition of $\psi_{a,X}(x) : h_a \rightarrow X$, defined for $b \in A$ as follows.

$$\begin{aligned} (\psi_{a,X}(x))_b : h_a(b) &\rightarrow X(b) \\ (\varphi : b \rightarrow a) &\mapsto X(\varphi)(x) \end{aligned}$$

We still need to check that $\psi_{a,X}(x) : h_a \rightarrow X$ is indeed a natural transformation. To do so let $\gamma : c \rightarrow b$ be a map in A , and consider the square

$$\begin{array}{ccc} h_a(b) & \xrightarrow{(\psi_{a,X}(x))_b} & X(b) \\ h_a(\gamma) \downarrow & & \downarrow X(\gamma) \\ h_a(c) & \xrightarrow{(\psi_{a,X}(x))_c} & X(c) \end{array}$$

We need to prove that it is commutative. Let $\varphi: b \rightarrow a$ be an element in $h_a(b)$. The top map sends it to $X(\varphi)(x)$, and composing with the right map gives $X(\gamma)(X(\varphi)(x)) = X(\varphi \circ \gamma)(x)$ (careful with the $(-)^{\text{op}}$). On the other hand, applying the left map to φ gives the precomposition with γ , $\varphi \circ \gamma$. The bottom map then sends this to $X(\varphi \circ \gamma)(x)$, and we have proven commutativity. Hence, $\psi_{a,X}(x)$ is indeed a map in $\hat{A}$ from h_a to X , and we can define the inverse to $\phi_{a,X}$,

$$\begin{aligned} \psi_{a,X}: X(a) &\rightarrow \text{Hom}_{\hat{A}}(h_a, X) \\ x &\mapsto \psi_{a,X}(x) \end{aligned}$$

To see that it is an inverse, we just compute, for $x \in X(a)$,

$$\begin{aligned} \phi_{a,X}(\psi_{a,X}(x)) &= (\psi_{a,X}(x))_a(\text{Id}_a) \\ &= X(\text{Id}_a)(x) \\ &= \text{Id}_{X(a)}(x) \\ &= x \end{aligned}$$

And on the other hand, for $f: h_a \rightarrow X$ in $\hat{A}$, we need to compare f and

$$\psi_{a,X}(\phi_{a,X}(f)) = \psi_{a,X}(f_a(\text{Id}_a)).$$

Since they are natural transformation, we need to show that for any b in A , we have the equality

$$f_b = (\psi_{a,X}(f_a(\text{Id}_a)))_b. \quad (5.1)$$

Now, by definition, the latter sends an element of $h_a(b)$, $\varphi: b \rightarrow a$ to the element of $X(b)$, $X(\varphi)(f_a(\text{Id}_a))$. But now since f is by assumption a natural transformation, we may consider the following commutative diagram.

$$\begin{array}{ccc} h_a(a) & \xrightarrow{f_a} & X(a) \\ h_a(\varphi) \downarrow & & \downarrow X(\varphi) \\ h_a(b) & \xrightarrow{f_b} & X(b) \end{array}$$

Starting with Id_a and then going through the left map gives φ , which then gives $f_b(\varphi) \in X(b)$. On the other hand, going through the top map first gives $f_a(\text{Id}_a)$, and applying $X(\varphi)$ gives $X(\varphi)(f_a(\text{Id}_a))$. Commutativity gives $f_b(\varphi) = X(\varphi)(f_a(\text{Id}_a))$, which finally gives the equality 5.1. Thus, we have proven that $\phi_{a,X}$ is a bijection with inverse $\psi_{a,X}$ for all object a in A and X in $\hat{A}$.

Now, we prove that $\phi_{a,X}$ is natural in X . Let $g: X \rightarrow Y$ be a map in $\hat{A}$, and a be an object of A . Consider the square

$$\begin{array}{ccc} \text{Hom}_{\hat{A}}(h_a, X) & \xrightarrow{\phi_{a,X}} & X(a) \\ \text{Hom}_{\hat{A}}(h_a, g) \downarrow & & \downarrow g_a \\ \text{Hom}_{\hat{A}}(h_a, Y) & \xrightarrow{\phi_{a,Y}} & Y(a) \end{array}$$

We need to prove that it is commutative. Start with $f: h_a \rightarrow X$ in the top left corner. Going through the left map gives $g \circ f: h_a \rightarrow Y$, and $\phi_{a,Y}(g \circ f) = (g \circ f)_a(\text{Id}_a)$. On the other hand, going through the top map first gives $f_a(\text{Id}_a)$, and then applying g_a gives $g_a(f_a(\text{Id}_a)) = (g \circ f)_a(\text{Id}_a)$. We have proven that the square is commutative, and thus that $\phi_{a,X}$ is natural in X .

Naturality of $\phi_{a,X}$ in a remains to be proven. Let X be an object in $\widehat{A}$ and $\varphi: b \rightarrow a$ be a map in A . Consider the square

$$\begin{array}{ccc} \text{Hom}_{\widehat{A}}(h_a, X) & \xrightarrow{\phi_{a,X}} & X(a) \\ \text{Hom}_{\widehat{A}}(h_\varphi, X) \downarrow & & \downarrow X(\varphi) \\ \text{Hom}_{\widehat{A}}(h_b, X) & \xrightarrow{\phi_{b,X}} & X(b) \end{array}$$

Starting with $f: h_a \rightarrow X$ and going through the top and right arrows gives $X(\varphi)(f_a(\text{Id}_a))$. On the other hand, applying the left arrow first gives $f \circ h_\varphi$, which is sent to $(f \circ h_\varphi)_b(\text{Id}_b)$. Now, to conclude, we need to prove the equality

$$X(\varphi)(f_a(\text{Id}_a)) = (f \circ h_\varphi)_b(\text{Id}_b).$$

Observe that the right hand side is equal to $f_b(h_\varphi(\text{Id}_b)) = f_b(\varphi)$, and we have already seen that $f_b(\varphi) = X(\varphi)(f_a(\text{Id}_a))$, which proves the equality. $\square$

5.2.3 Corollaries and remarks

Let us list a few classical corollaries of the Yoneda lemma.

Corollary 5.2.4. *Let A be a small category. The Yoneda embedding $y: A \rightarrow \widehat{A}$ is fully-faithful. i.e. it is an embedding.*

Proof. Let a and b be two objects of A , and consider the map induced by the Yoneda embedding

$$\begin{aligned} y_{a,b}: \text{Hom}_A(a, b) &\rightarrow \text{Hom}_{\widehat{A}}(h_a, h_b) \\ \varphi &\mapsto h_\varphi: h_a \rightarrow h_b \end{aligned}$$

Using Theorem 5.2.3, we also get a bijection

$$\begin{aligned} \phi_{a,h_b}: \text{Hom}_{\widehat{A}}(h_a, h_b) &\rightarrow h_b(a) = \text{Hom}_A(a, b) \\ f: h_a &\rightarrow h_b \mapsto f_a(\text{Id}_a) \end{aligned}$$

Now, given a map $\varphi: a \rightarrow b$ in A , we may compute the composition

$$\begin{aligned} \phi_{a,h_b} \circ y_{a,b}(\varphi) &= \phi_{a,h_b}(h_\varphi) \\ &= (h_\varphi)_a(\text{Id}_a) \\ &= \varphi \end{aligned}$$

hence $\phi_{a,h_b} \circ y_{a,b} = \text{Id}$. Since we already know that ϕ_{a,h_b} is a bijection, this means that $y_{a,b}$ is a bijection, hence that y is fully-faithful. $\square$

Corollary 5.2.5. *Let A be a small category and a and b be two objects of A . Assume that there exists a natural transformation*

$$\mathrm{Hom}_A(-, a) \simeq \mathrm{Hom}_A(-, b)$$

i.e. and isomorphism in $\widehat{A}$ $h_a \simeq h_b$. Then, there exists an isomorphism in A between a and b . The same is true if $\mathrm{Hom}_A(a, -)$ and $\mathrm{Hom}_A(b, -)$ are isomorphic.

Proof. Let $f: h_a \rightarrow h_b$ be the isomorphism we assumed existed and let $f^{-1}: h_b \rightarrow h_a$ be its inverse. Since $y_{a,b}: \mathrm{Hom}_A(a, b) \rightarrow \mathrm{Hom}_{\widehat{A}}(h_a, h_b)$ is a bijection, there exists $\varphi: a \rightarrow b$ such that $f = h_\varphi$. Similarly, there exists $\varphi^{-1}: b \rightarrow a$ such that $f^{-1} = h_{\varphi^{-1}}$. But then, we have

$$y(\varphi \circ \varphi^{-1}) = f \circ f^{-1} = \mathrm{Id}_{h_b} = y(\mathrm{Id}_b)$$

And injectivity of $y_{b,b}$ implies that $\varphi \circ \varphi^{-1} = \mathrm{Id}_b$. Similarly, we get that $\varphi^{-1} \circ \varphi = \mathrm{Id}_a$, and that $a \simeq b$. The dual claim follows by considering A^{op} and $\widehat{A^{\mathrm{op}}}$. $\square$

Remark 5.2.6. We have already seen a more general version of Corollary 5.2.5. This was Lemma 3.3.2. In fact, the proof of Lemma 3.3.2 is arguably simpler, since it does not depend on the full proof of the Yoneda lemma. However, it is extremely common to present this claim (that representable functors detect isomorphisms) as a corollary of Yoneda lemma. And while it is true that when Yoneda lemma holds, this corollary holds too, we often need the corollary for categories $\mathcal{C}$ that may not be small, and thus the Yoneda lemma does not apply. This is the reason why we proved it by hand, and delayed so much the introduction of the Yoneda lemma. If one is rigorous with the distinction between small and non-small category, then the Yoneda lemma appears for what it is: a fundamental statement *about categories of presheaves*. And this "corollary" is rather a general statement about categories, that happen to reduce to a particular case of the Yoneda lemma *for small categories*. Of course this distinction becomes entirely pedantic if one works in a setting in which any category can be assumed to be small. There is a pedagogical reason to present both those claims together, and it is that if one tries to prove the corollary by hand, (as we did in the proof of Lemma 3.3.2), the arguments used are essentially the same as those appearing in the proof of Theorem 5.2.3.

Remark 5.2.7. To contrast with the previous remark, the Yoneda lemma is absolutely central when working with presheaves. Indeed, from now on, when a is an object of a small category A and X an object of $\widehat{A}$, we will constantly go back and forth between elements of the set $X(a)$ and maps $h_a \rightarrow X$, and we will not be concerned with distinguishing between the two. This should not be understood as a claim that there is a single bijection between the sets $X(a)$ and $\mathrm{Hom}_{\widehat{A}}(h_a, X)$, (indeed, there are many in general), but as a convention that unless explicitly specified otherwise, we will only consider the bijection introduced in Theorem 5.2.3.

We conclude with the following proposition. Its claim is entirely contained in the statement of the Yoneda lemma, but it may be useful at first to think of it as an independent statement, to work through the coming proofs.

Proposition 5.2.8. *Let A be a small category, X, Y presheaves on A , $f: X \rightarrow Y$ a map in $\hat{A}$, a, b objects of A , $\varphi: b \rightarrow a$ a map in A , and $\sigma: h_a \rightarrow X$ and $\tau: h_b \rightarrow X$ be two maps in $\hat{A}$. Then the following diagram is commutative.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \swarrow \sigma & \nearrow f_a(\sigma) \\ & h_a & \end{array}$$

Furthermore, the following diagram is commutative if and only if $X(\varphi)(\sigma) = \tau$.

$$\begin{array}{ccc} & X & \\ \tau \nearrow & & \nwarrow \sigma \\ h_b & \xrightarrow{h_\varphi} & h_a \end{array}$$

Proof. This follows from the naturality of the bijection of Theorem 5.2.3, together with the implicit identification under this bijection, addressed in Remark 5.2.7. $\square$

5.3 Comma categories and categories of presheaves

In this section, we discuss several construction of the same flavor that are meaningful on categories of presheaves. All are examples of comma categories, so we take some time to introduce the more general notion first.

5.3.1 Comma categories

Definition 5.3.1. *Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{E} \rightarrow \mathcal{D}$ be two functors. The comma category of F and G , denoted $F \downarrow G$ is defined as follows.*

- its objects are tuples (X, Y, α) , where X and Y are objects of $\mathcal{C}$ and $\mathcal{E}$ respectively, and $\alpha: F(X) \rightarrow G(Y)$ is a map in $\mathcal{D}$.
- its morphisms are pairs $(f, g): (X, Y, \alpha) \rightarrow (X', Y', \alpha')$ where $f: X \rightarrow X'$, and $g: Y \rightarrow Y'$ are maps in $\mathcal{C}$ and $\mathcal{E}$ respectively. such that the following square is commutative.

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(X') \\ \alpha \downarrow & & \downarrow \alpha' \\ G(Y) & \xrightarrow{G(g)} & G(Y') \end{array}$$

- *Compositions and identities are defined from those of $\mathcal{C}$ and $\mathcal{E}$.*

Remark 5.3.2. Observe that the comma category $F \downarrow G$ comes equipped with two projection functors $P_{\mathcal{C}}: F \downarrow G \rightarrow \mathcal{C}$ and $P_{\mathcal{E}}: F \downarrow G \rightarrow \mathcal{E}$. Sending (X, Y, α) to X and Y respectively. Furthermore, observe that the compositions $F \circ P_{\mathcal{C}}$ and $G \circ P_{\mathcal{E}}$ are both functors $F \downarrow G \rightarrow \mathcal{D}$ but they are not isomorphic in general. In fact, they are related by a (tautological) natural transformation, $\gamma: F \circ P_{\mathcal{C}} \rightarrow G \circ P_{\mathcal{E}}$ defined as follows, for an object (X, Y, α) in $F \downarrow G$,

$$\gamma_{(X,Y,\alpha)} = \alpha: F(X) \rightarrow G(Y).$$

The entire situation is summarized in the following (non-commutative) diagram.

$$\begin{array}{ccc} F \downarrow G & \xrightarrow{P_{\mathcal{E}}} & \mathcal{E} \\ P_{\mathcal{C}} \downarrow & \nearrow \gamma & \downarrow G \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

We see that the square of functors fails to be commutative, but instead of commutation, we get the natural transformation γ relating the two compositions.

The definition of a comma category is rarely useful in its full generality. However, it admits several interesting specialization. Let us consider a few.

Example 5.3.3. When $\mathcal{C} = \mathcal{D}$, $\mathcal{E}$ is the category discrete category with a single object, a functor $G: \mathcal{E} \rightarrow \mathcal{C}$ then amounts to the choice of some object Y in $\mathcal{C}$. The comma category $\text{Id}_{\mathcal{C}} \downarrow G$ then gives back the overcategory, first described in Example 4.1.10, and which reappeared in Exercise 63 and Example 5.1.7. It is denoted $\mathcal{C}/Y$, and if we want to recall its definition as a comma category, it will typically be denoted $\text{Id}_{\mathcal{C}} \downarrow Y$ or even just $\mathcal{C} \downarrow Y$.

Example 5.3.4. This time, take $\mathcal{D} = \mathcal{E}$, and $\mathcal{C}$ to be the discrete category with one object. Once again, picking a functor $\mathcal{C} \rightarrow \mathcal{D}$ amounts to choosing an object X in $\mathcal{D}$, thus we may identify the functor $F: \mathcal{C} \rightarrow \mathcal{D}$ with its image. The comma category $X \downarrow \text{Id}_{\mathcal{D}}$ is now the undercategory. (or coslice category). Its objects are pairs (Y, f) , where Y is an object in $\mathcal{D}$ and $f: X \rightarrow Y$, is a map in $\mathcal{D}$, and morphisms are commutative triangles

$$\begin{array}{ccc} & X & \\ f \swarrow & & \searrow g \\ Y & \xrightarrow{h} & Z \end{array}$$

It may be denoted $X \downarrow \mathcal{D}$ or $X \backslash \mathcal{D}$, though the former is preferable. Note that $X \downarrow \mathcal{D}$ is equivalent to $(\mathcal{D}^{\text{op}} \downarrow X)^{\text{op}}$, that is overcategory and undercategory are dual notions.

Example 5.3.5. We will also consider slightly more general examples, where $F: \mathcal{C} \rightarrow \mathcal{D}$ is a non-identity functor, and $G: \mathcal{E} \rightarrow \mathcal{D}$ is a functor from the discrete category with a single object. (i.e. correspond to some object Y in $\mathcal{D}$). In that

case, objects of the comma category $F \downarrow Y$ are pairs (X, f) , where X is an object of $\mathcal{C}$ and $f: F(X) \rightarrow Y$ is a map in $\mathcal{D}$, and morphisms $(X, f) \rightarrow (X', f')$ are maps $g: X \rightarrow X'$ such that the triangle

$$\begin{array}{ccc} F(X) & \xrightarrow{F(g)} & F(X') \\ & \searrow f & \swarrow f' \\ & Y & \end{array}$$

Exercise 66. Let $\mathcal{D}$ be a category, and X and Y be two objects of $\mathcal{D}$ (We will also write X and Y for the corresponding functors from the discrete category with one object). Let $\mathcal{C}$ be the full subcategory of $\mathcal{D}$ containing only the object X and $\mathcal{E}$ be the full subcategory of $\mathcal{D}$ containing only the object Y . Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{E} \rightarrow \mathcal{D}$ be the inclusions of the two subcategories.

- Show that the four categories

- $X \downarrow Y$
- $F \downarrow Y$
- $X \downarrow G$
- $F \downarrow G$

Are all small categories with set of objects $\text{Hom}_{\mathcal{D}}(X, Y)$

- Show that the categories above have distinct sets of morphisms.

5.3.2 Categories of elements

Definition 5.3.6. Let A be a small category, and X an object of $\hat{A}$. The **category of elements** of X , denoted A/X is the comma category $y \downarrow X$ where $y: A \rightarrow \hat{A}$ is the yoneda embedding.

Remark 5.3.7. The above definition may seem opaque, but it can be made quite explicit. By definition, an object of A/X is the data of an object a of A and a map $\sigma: y(a) = h_a \rightarrow X$. By Theorem 5.2.3, this is equivalent to the data of $\sigma \in X(a)$. Thus, A/X is a small category with set of objects the disjoint union $\coprod_{a \in \text{obj}(A)} X(a)$, i.e. the set of *elements* of X . Similarly, morphisms in A/X from $(a, \sigma) \rightarrow (b, \tau)$ are maps $\varphi: a \rightarrow b$ such that the following triangle commutes

$$\begin{array}{ccc} h_a & \xrightarrow{h_\varphi} & h_b \\ & \searrow \sigma & \swarrow \tau \\ & X & \end{array}$$

Working through the yoneda lemma, commutativity of this diagram is equivalent to the claim that $X(\varphi)(\tau) = \sigma$. In general, for $\sigma \in X(a)$ and $\tau \in X(b)$, we have

$$\text{Hom}_{A/X}(\sigma, \tau) = \{\varphi: a \rightarrow b \mid X(\varphi)(\tau) = \sigma\}$$

One of the important reasons to care about the category of elements is that it contains all of the information about the presheaf X in a way that can be easily recovered.

Proposition 5.3.8. *Let X be a presheaf on A . Let $D: A/X \rightarrow \hat{A}$ be the composition of the projection $P_A: A/X \rightarrow A$ with the yoneda embedding $A \rightarrow \hat{A}$. The collection of maps*

$$\sigma: h_a = D(a, \sigma) \rightarrow X \quad (5.2)$$

forms a colimit cocone. i.e.

$$\lim_{\rightarrow} D \simeq X$$

Proof. First, let us show that the collection of maps described in (5.2) indeed forms a cocone. It is a collection of maps indexed by objects (a, σ) of A/X , from $D(a, \sigma) \rightarrow X$, thus the only thing that needs to be checked is the commutativity condition. Let $\varphi: (a, \sigma) \rightarrow (b, \tau)$ be a map in A/X . Then, by definition of A/X , the following triangle commutes.

$$\begin{array}{ccc} h_a & \xrightarrow{h_\varphi} & h_b \\ & \searrow \sigma & \swarrow \tau \\ & X & \end{array}$$

But observe that it is also the triangle whose commutativity needs to be checked for those maps to form a cocone. Hence, we indeed have a cocone on X .

Let us prove that it is a colimit cocone. Let Y be a presheaf on A and let

$$f_{(a, \sigma)}: h_a \rightarrow Y$$

be a collection of maps in $\hat{A}$, forming a cocone on D . Observe that each of the maps $f_{(a, \sigma)}$ is an element of $Y(a)$. We may define a map in $\hat{A}$, $\tilde{f}: X \rightarrow Y$, defined for objects a of A as the map

$$\begin{aligned} \tilde{f}_a: X(a) &\rightarrow Y(a) \\ \sigma &\mapsto f_{(a, \sigma)} \end{aligned}$$

We need to prove that $\tilde{f}$ is indeed a map in $\hat{A}$, i.e. that it is a natural transformation between the functors X and Y . To do so, let $\varphi: b \rightarrow a$ be a map in A , and consider the square

$$\begin{array}{ccc} X(a) & \xrightarrow{\tilde{f}_a} & Y(a) \\ X(\varphi) \downarrow & & \downarrow Y(\varphi) \\ X(b) & \xrightarrow{\tilde{f}_b} & Y(b) \end{array}$$

Starting with an element $\sigma \in X(a)$, the top right composition gives the element $Y(\varphi)(f_{(a, \sigma)})$, while the left bottom composition gives $f_{(b, X(\varphi)(\sigma))}$. But now,

recall that the collection of $f_{(a,\sigma)}$ formed a cocone. Since φ gives a map in A/X from $(b, X(\varphi)(\sigma))$ to (a, σ) , we must have a commutative diagram

$$\begin{array}{ccc} h_b & \xrightarrow{h_\varphi} & h_a \\ & \searrow f_{(b, X(\varphi)(\sigma))} & \swarrow f_{(a,\sigma)} \\ & Y & \end{array}$$

But now the Yoneda lemma (more specifically, naturality in the first variable) tells us that commutativity of this diagram implies the equality $Y(\varphi)(f_{(a,\sigma)}) = f_{(b, X(\varphi)(\sigma))}$. This is the desired equality, and we deduce that $\tilde{f}: X \rightarrow Y$ is indeed a map in $\hat{A}$. (Note that from now on, whenever such a map is to appear, we will simply denote it f .)

Now that we now that $\tilde{f}$ is a map of presheaves, let us prove that it factors the cocone. Consider the diagram

$$\begin{array}{ccc} & X & \\ \sigma \nearrow & & \searrow \tilde{f} \\ h_a & \xrightarrow{f_{(a,\sigma)}} & Y \end{array} \quad (5.3)$$

By construction, $\tilde{f} \circ \sigma = \tilde{f}_a(\sigma) = f_{(a,\sigma)}$ where the first (and last) equality are to be understood under the identification allowed by the Yoneda lemma.

Unicity of the factorization remains to be proven. Assume that $g: X \rightarrow Y$ is some other map in $\hat{A}$ making the diagram (5.3) commute. Then, commutativity of the diagram impose that $g_a(\sigma) = f_{(a,\sigma)}$ for all $a \in \text{obj}(A)$ and $\sigma \in X(a)$, but this precisely means that $\tilde{f} = g$. Which concludes the proof. $\square$

Remark 5.3.9. The proof above may seem obscure, at least at first glance. Once one get used to the Yoneda, it should appear for what it is: a tedious back and fourth through the Yoneda lemma, but containing no new ideas.

Exercise 67. Define a functor $\text{El}: \hat{A} \rightarrow \underline{\text{Cat}}$ sending each object X of $\hat{A}$ to its category of elements A/X .

Exercise 68. Let $f: X \rightarrow Y$ be a map in the category of presheaves $\hat{A}$. Let $D_X: A/X \rightarrow \hat{A}$ and $D_Y: A/Y \rightarrow \hat{A}$ be the functors of Proposition 5.3.8. Let $\alpha_{(a,\sigma)}: D_X(a, \sigma) \rightarrow X$, for $(a, \sigma) \in \text{obj}(A/X)$, and $\beta_{(a,\tau)}: D_Y(a, \tau) \rightarrow Y$ for $(a, \tau) \in \text{obj}(A/Y)$ be the corresponding colimit cones.

1. Show that the collection $\beta_{(a, f_a(\sigma))} \circ \text{Id}_{h_a}: D_X(a, \sigma) \rightarrow Y$, for $(a, \sigma) \in \text{obj}(A/X)$ forms a cocone on D_X .
2. Show that map $X \simeq \lim_{\rightarrow} D_X \rightarrow Y$ induced by the universal property of the colimit is equal to $\tilde{f}: X \rightarrow Y$.

5.3.3 Overcategories of categories of presheaves

We conclude this section with the following useful observation.

Proposition 5.3.10. *Let A be a small category, and X a presheaf on A . There is an equivalence of categories.*

$$(\widehat{A})/X \simeq \widehat{(A/X)}$$

Proof. We will define the functors by hand.

Let (Y, f) be an object of $(\widehat{A})/X$. i.e. Y is a presheaf on A and $f: Y \rightarrow X$ is a map of presheaves. We define $F: (\widehat{A})/X \rightarrow \widehat{(A/X)}$ on objects by sending (Y, f) to the presheaf:

$$\begin{aligned} F(Y, f): (A/X)^{\text{op}} &\rightarrow \underline{\text{Set}} \\ (a, \sigma) &\mapsto (f_a)^{-1}(\sigma) \subset Y(a) \end{aligned}$$

We also need to define what $F(Y, f)$ does to maps in A/X . Let $\varphi: (b, \tau) \rightarrow (a, \sigma)$ be a map in A/X . Then we have the map of sets $Y(\varphi): Y(a) \rightarrow Y(b)$. Furthermore, since $f: Y \rightarrow X$ is a map of presheaves, the following diagram must commute.

$$\begin{array}{ccc} Y(a) & \xrightarrow{f_a} & X(a) \\ Y(\varphi) \downarrow & & \downarrow X(\varphi) \\ Y(b) & \xrightarrow{f_b} & X(b) \end{array}$$

Now, by assumption $X(\varphi)(\sigma) = \tau$, this shows that $Y(\varphi)(f_a^{-1}(\sigma)) \subset f_b^{-1}(\tau)$. Thus, we can define $F(Y, f)(\varphi) = Y(\varphi)|_{f_a^{-1}(\sigma)}^{f_b^{-1}(\tau)}: F(Y, f)(a, \sigma) \rightarrow F(Y, f)(b, \tau)$. This defines the presheaf $F(Y, f)$ on A/X . Now, assume that $h: (Y, f) \rightarrow (Z, g)$ is a map in $(\widehat{A})/X$. Then, by definition of the maps in an overcategory, we must have $h(f_a^{-1}(\sigma)) \subset g_a^{-1}(\sigma)$ for all objects (a, σ) of A/X . Hence, we may define a map of presheaves $F(h): F(Y, f) \rightarrow F(Z, g)$ via, for all objects (a, σ) of A/X ,

$$F(h)_{(a, \sigma)} = (h_a)|_{f_a^{-1}(\sigma)}^{g_a^{-1}(\sigma)}: F(Y, f)(a, \sigma) \rightarrow F(Z, g)(a, \sigma)$$

Exercise 69. Check that the functor $F: (\widehat{A})/X \rightarrow \widehat{(A/X)}$ is well-defined.

For the other direction, let $Y: (A/X)^{\text{op}} \rightarrow \underline{\text{Set}}$ be a presheaf on A/X . We need to define an object $G(Y)$ of $(\widehat{A})/X$. We will define separately an object of $\widehat{A}$, $G_1(Y)$, and a morphism of $\widehat{A}$, $G_2(Y): G_1(Y) \rightarrow X$. Let a be an object of A , and set

$$G_1(Y)(a) = \coprod_{\sigma \in X(a)} Y(a, \sigma)$$

Now, given a map in A , $\varphi: b \rightarrow a$, for each $\sigma \in X(a)$, there is a map $\varphi: (b, \tau) \rightarrow (a, \sigma)$ in A/X , with $\tau = X(\varphi)(\sigma)$. Hence, we have maps $Y(\varphi): Y(a, \sigma) \rightarrow$

$Y(b, \tau)$. Those give a map

$$G_1(Y)(\varphi): \coprod_{\sigma \in X(a)} Y(a, \sigma) \rightarrow \coprod_{\tau \in X(b)} Y(b, \tau)$$

where each component in the disjoint union in the domain is sent to exactly one component in the disjoint union in the codomain. Now, for a an object of A , define $G_2(Y)_a: G_1(Y)(a) \rightarrow X(a)$ as the map sending every element in the component $Y(a, \sigma) \subset \coprod_{\tau \in X(a)} Y(a, \tau)$ to $\sigma \in X(a)$. Naturality then follows from the definition of $G_1(Y)$.

Exercise 70. Check that $(G_1(Y), G_2(Y))$ is a well-defined object of $(\widehat{A})/X$.

Now, assume that $f: Y \rightarrow Z$ is a map in $(\widehat{A/X})$. Define $G(f): G_1(Y) \rightarrow G_1(Z)$ as the map with value at an object a of A :

$$G(f)_a = \coprod_{\sigma \in X(a)} f_{(a, \sigma)}: \coprod_{\sigma \in X(a)} Y(a, \sigma) \rightarrow \coprod_{\sigma \in X(a)} Z(a, \sigma)$$

This gives a functor $G: (\widehat{A/X}) \rightarrow (\widehat{A})/X$.

Let us now check that F and G are inverse equivalences. Let (Y, f) be an object of $(\widehat{A})/X$, we compute, and let a be an object of A . We compute

$$\begin{aligned} (G_1 \circ F(Y, f))(a) &= \coprod_{\sigma \in X(a)} F(Y, f)(a, \sigma) \\ &= \coprod_{\sigma \in X(a)} f_a^{-1}(\sigma) \\ &\simeq Y(a) \end{aligned}$$

Where the last isomorphism follows from the observation that $Y(a) = f_a^{-1}(X(a))$, hence $Y(a)$ is a model for the disjoint union of the $f_a^{-1}(\sigma)$, for $\sigma \in X(a)$. To make the isomorphism explicit, we may write an element in the disjoint union $\coprod_{\sigma \in X(a)} f_a^{-1}(\sigma)$ as a pair (δ, σ) , where $\delta \in f_a^{-1}(\sigma)$. The isomorphism then sends (δ, σ) to $\delta \in Y(a)$. This gives an explicit isomorphism $(\alpha_{(Y, f)})_a: (G_A \circ F(Y, f))(a) \rightarrow Y(a)$. Note that the collection of $(\alpha_{(Y, f)})_a$ indexed by objects a of A gives an isomorphism in $\widehat{A}$, $\alpha_{(Y, f)}: G_1 \circ F(Y, f) \rightarrow Y$. To see that $\alpha_{(Y, f)}$ is a natural transformation (i.e. a morphism in $\widehat{A}$), let $\varphi: b \rightarrow a$ be a morphism in A , and consider the square

$$\begin{array}{ccc} \coprod_{\sigma \in X(a)} f_a^{-1}(\sigma) & \xrightarrow{(\alpha_{(Y, f)})_a} & Y(a) \\ \Pi \left(Y(\varphi) \Big|_{f_a^{-1}(\sigma)}^{f_b^{-1}(X(\varphi)(\sigma))} \right) \downarrow & & \downarrow Y(\varphi) \\ \coprod_{\tau \in X(b)} f_b^{-1}(\tau) & \xrightarrow{(\alpha_{(Y, f)})_b} & Y(b) \end{array}$$

Starting with a pair (δ, σ) in the top left corner, both composition give $Y(\varphi)(\delta)$, hence the square commutes, and we have proven that $\alpha_{(Y,f)}$ is indeed an isomorphism in $\widehat{A}$. Let us now show that it gives an isomorphism in $(\widehat{A})/X$, i.e. that the following triangle commutes.

$$\begin{array}{ccc} G_1 \circ F(Y, f) & \xrightarrow{\alpha_{Y,f}} & Y \\ & \searrow & \swarrow f \\ & X & \end{array}$$

Let a be an object of A and (δ, σ) be an element of $G_1 \circ F(Y, f)(a)$. i.e. $\sigma \in X(a)$ and $\delta \in f_a^{-1}(\sigma)$. Then, $G_2 \circ F(Y, f)(\delta, \sigma) = \sigma = f(\delta) = f \circ \alpha_{(Y,f)}(\delta, \sigma)$, which proves commutativity of the triangle. Hence, we have proven that $\alpha_{(Y,f)}: G \circ F(Y, f) \rightarrow (Y, f)$ is an isomorphism in $(\widehat{A})/X$. It remains to be proven that $\alpha: G \circ F \rightarrow \text{Id}_{(\widehat{A})/X}$ is a natural transformation. Let $h: (Y, f) \rightarrow (Z, g)$ be a map in $(\widehat{A})/X$, and consider the square

$$\begin{array}{ccc} G \circ F(Y, f) & \xrightarrow{\alpha_{(Y,f)}} & (Y, f) \\ G \circ F(h) \downarrow & & \downarrow h \\ G \circ F(Z, g) & \xrightarrow{\alpha_{(Z,g)}} & (Z, g) \end{array}$$

Starting with an element $(\delta, \sigma) \in G \circ F(Y, f)(a)$ in the top left corner, the left arrow gives $(h(\delta), \sigma)$ and applying $\alpha_{(Z,g)}$ gives $h(\delta)$. On the other hand, going through the top arrow first and then the right gives δ and then $h(\delta)$, hence the diagram is commutative, and we have exhibited a natural isomorphism $\alpha: G \circ F \simeq \text{Id}_{(\widehat{A})/X}$.

To complete the proof, one also needs to exhibit a natural isomorphism $F \circ G \simeq \text{Id}_{(\widehat{A/X})}$. We leave this part as an exercise.

Exercise 71. Show that there exists a natural isomorphism $F \circ G \simeq \text{Id}_{(\widehat{A/X})}$. $\square$

Remark 5.3.11. The above proof may seem complicated, but the difficulty essentially stems from unfolding every definition. The key idea is quite simple. The data of a map $f: Y \rightarrow X$ is equivalent to the data of the fibers $f^{-1}(\sigma) \subset Y$, for all $\sigma \in X$. The proof then repeatedly uses this observation, but on sets of elements of presheaves, $Y(a)$ and $X(a)$ for $a \in \text{obj}(A)$.

Remark 5.3.12. The above proposition illustrates a key fact about categories of presheaves. When one works in a category of presheaves, it is easy to stay in the context of categories of presheaves. As we have just seen slice categories in categories of presheaves are themselves categories of presheaves. We may also show that if A and B is a small category, then $\text{Fun}(A, \widehat{B})$ is a category of presheaves.

Exercise 72. Show that $\text{Fun}(A^{\text{op}}, \widehat{B}) \simeq (\widehat{A \times B})$

This is one reason to care about category of presheaves. They enjoy nice properties, and if one successfully reduces the study of some category of interest to that of a category of presheaves, then those nice properties will stay true even if one goes through extra constructions.

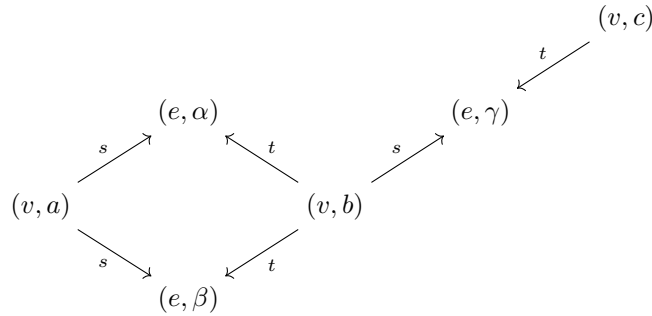
Exercise 73. Reinterpret Exercises 5.1.6 and 5.1.7 as applications of Proposition 5.3.10.

Finally, let us summarize the constructions in this section with a complete example.

Example 5.3.13. Consider the graph G with set of vertices $V(G) = \{a, b, c\}$ and set of edges $\{\alpha, \beta, \gamma\}$, where $s_G(\alpha) = s_G(\beta) = a$, $t_G(\alpha) = t_G(\beta) = s_G(\gamma) = b$ and $t_G(\gamma) = c$. It is represented in Figure 5.1. Let A be the small category whose presheaves are graphs. i.e. A has two objects v and e and two non-identity morphisms $s, t: v \rightarrow e$. The graph G is an object of $\hat{A}$ (see Example 5.1.5), with $G(v) = V(G)$ and $G(e) = E(G)$. We may compute A/G . By Remark 5.3.7, its set of objects is in bijection with the disjoint union $\text{obj}(A/G) = G(v) \amalg G(e)$. To be consistent with the notations we used so far, we will denote the objects as follows $\text{obj}(A/G) = \{(v, a), (v, b), (v, c), (e, \alpha), (e, \beta), (e, \gamma)\}$. Morphisms in A/G are obtained from those of A . Since the only non-identity morphisms in A go from v to e , the same will be true in A/G . Furthermore, we will have a morphism $s: (v, x) \rightarrow (e, y)$ for $x \in \{a, b, c\}$ and $y \in \{\alpha, \beta, \gamma\}$ if and only if $G(s)(y) = s_G(y) = x$. And similarly for t . Thus, the non identity morphisms in A/G are the elements of the following list.

- $s: (v, a) \rightarrow (e, \alpha)$
- $s: (v, a) \rightarrow (e, \beta)$
- $s: (v, b) \rightarrow (e, \gamma)$
- $t: (v, b) \rightarrow (e, \alpha)$
- $t: (v, b) \rightarrow (e, \beta)$
- $t: (v, c) \rightarrow (e, \gamma)$

Thus, all non-identity morphisms in the category A/G can be represented in the following diagram



Now, Proposition 5.3.8 states that the diagram $D_G: A/G \rightarrow \widehat{A}$ admits G as its colimit. The diagram D_G is also represented in Figure 5.1.

Now, we may consider the category $\mathbf{Graph}/G$. By Proposition 5.3.10, it is equivalent to the category of presheaves over A/G . To see this, observe that an object of $\mathbf{Graph}/G$ is a graph $H = (E(H), V(H), s_H, t_H)$ with a map of graphs $f: H \rightarrow G$. This map of graph amounts to two maps $f_E: E(H) \rightarrow E(G) = \{\alpha, \beta, \gamma\}$ and $f_V: V(H) \rightarrow V(G) = \{a, b, c\}$, with the compatibility conditions that $s_G \circ f_E = f_V \circ s_H$ and $t_G \circ f_E = f_V \circ t_H$. Equivalently, we may specify f by recalling the subsets $(f_E)^{-1}(\alpha), (f_E)^{-1}(\beta), (f_E)^{-1}(\gamma) \subset E(H)$ and $(f_V)^{-1}(a), (f_V)^{-1}(b), (f_V)^{-1}(c) \subset V(H)$. Those sets correspond exactly to the values on objects of a functor $X: (A/G)^{\text{op}} \rightarrow \mathbf{Set}$, if one sets $X(v, a) = (f_V)^{-1}(a)$, and so on, for all objects in A/G . This corresponds to thinking of vertices in H as being of one of three possible kinds, a, b or c , and arrows in H of being of one of three kinds α, β or γ . Where, for example, an arrow of the kind β must have a source of the kind a and a target of the kind b . Now, we can either think of this as some extra data specified on a graph H , or think of such a structure as a different kind of graphs entirely (a specific kind of *presheaf* to be specific), which is the data of 6 sets, $X(a), X(b), X(c), X(\alpha), X(\beta), X(\gamma)$ (three kinds of vertices and three kinds of edges) and 6 maps, $X(s): X(\alpha) \rightarrow X(a)$, $X(s): X(\beta) \rightarrow X(a)$, $X(s): X(\gamma) \rightarrow X(b)$, $X(t): X(\alpha) \rightarrow X(b)$, $X(t): X(\beta) \rightarrow X(b)$ and $X(t): X(\gamma) \rightarrow X(c)$, (one source morphism and one target morphism for each kind of edge). But this second point of view corresponds exactly to the data of a presheaf on A/G .

5.4 Computations in categories of presheaves

As previously stated, the main motivation to study categories of presheaves is that they enjoy very good properties. In this section, we list two of those good properties. The first is that categories of presheaves are complete and cocomplete and that their (co)-limits are very easy to compute. The second is that functors between presheaf categories (or more generally from a presheaf category) are very easy to define, and in fact, often come in the form of adjoint pair for free.

5.4.1 Limits and colimits in presheaves

Definition 5.4.1. *Let A be a small category, and a be an object of A . We define the evaluation at a functor, ev_a , as follows.*

$$\begin{aligned} \text{ev}_a: \widehat{A} &\rightarrow \mathbf{Set} \\ X &\mapsto X(a) \\ f: X &\rightarrow Y \mapsto f_a: X(a) \rightarrow Y(a) \end{aligned}$$

Given a map $\varphi: b \rightarrow a$ in A , define the natural transformation $\text{ev}_\varphi: \text{ev}_a \rightarrow$

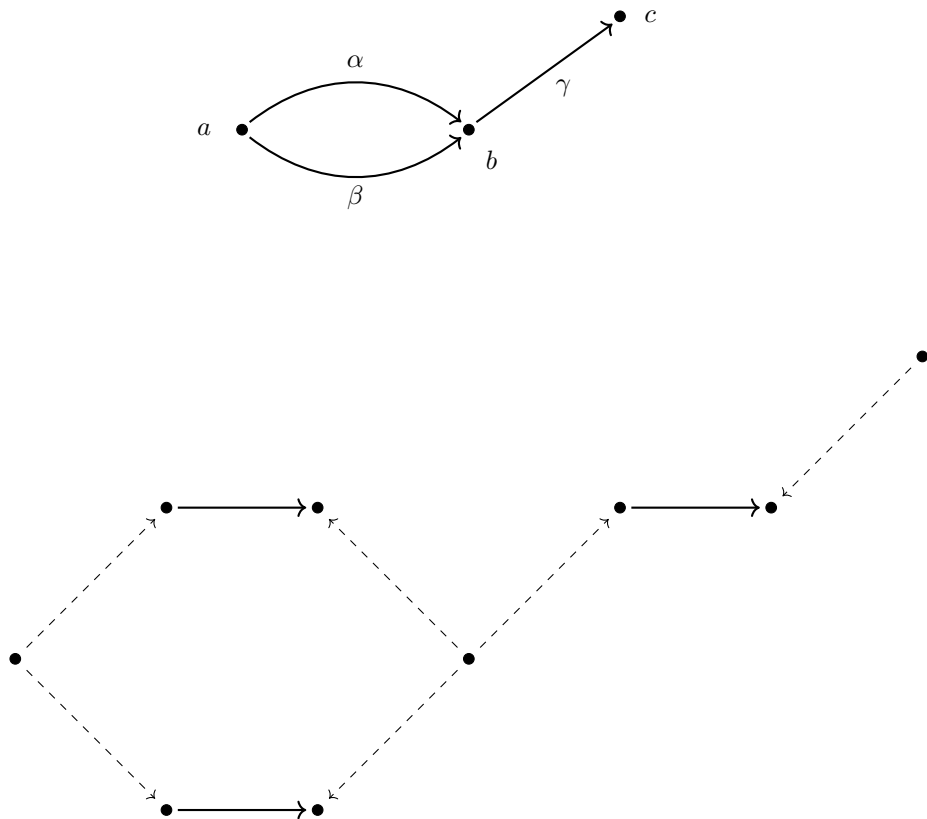


Figure 5.1: The directed graph, G of Example 5.3.13, and its associated diagram $D_G: A/G \rightarrow \mathbf{Graph}$ below. Morphisms in the diagram are represented by dashed arrows, while the solid arrows correspond to the representable graph associated to $e \in \mathbf{obj}(A)$.

ev_b , defined at an object X of $\widehat{A}$ as

$$(\text{ev}_\varphi)_X = X(\varphi): X(a) \rightarrow X(b)$$

Exercise 74. Check that $\text{ev}_\varphi: \text{ev}_a \rightarrow \text{ev}_b$ is a natural transformation, for any map $\varphi: b \rightarrow a$ in A .

Intuitively, the evaluation at a associates to a presheaf its set of a -elements. Now, a very important fact is that (co)-limits in the category of presheaves are computed **object-wise**, that is to say, they commute with the evaluation functors.

Theorem 5.4.2. *Let A be a small category, and $D: \mathcal{I} \rightarrow \widehat{A}$ be a diagram of presheaves. Then, $\lim_{\leftarrow} D$ and $\lim_{\rightarrow} D$ exist, and for any object a in A , we have*

$$\begin{aligned} \text{ev}_a(\lim_{\leftarrow} D) &\simeq \lim_{\leftarrow} (\text{ev}_a \circ D) \\ \lim_{\rightarrow} (\text{ev}_a \circ D) &\simeq \text{ev}_a(\lim_{\rightarrow} D) \end{aligned}$$

In order to make the proof of the above theorem clearer, let us state explicitly a simple but useful lemma.

Lemma 5.4.3. *Let $D, D': \mathcal{I} \rightarrow \mathcal{C}$ be two diagrams which admit limits, and $f: D \rightarrow D'$ be a natural transformation. Then there exists a unique map in $\mathcal{C}$,*

$$\lim_{\leftarrow} f: \lim_{\leftarrow} D \rightarrow \lim_{\leftarrow} D'$$

such that for any $i \in \text{obj}(\mathcal{I})$, the following diagram commutes.

$$\begin{array}{ccc} \lim_{\leftarrow} D & \xrightarrow{\lim_{\leftarrow} f} & \lim_{\leftarrow} D' \\ \downarrow & & \downarrow \\ D(i) & \xrightarrow{f_i} & D'(i) \end{array}$$

Where the vertical arrows are the arrows forming the limit cones on D and D' respectively.

Similarly for the colimits, there exists a unique map

$$\lim_{\rightarrow} f: \lim_{\rightarrow} D \rightarrow \lim_{\rightarrow} D'.$$

Proof. Let $\alpha_i: \lim_{\leftarrow} D \rightarrow D(i)$, for $i \in \text{obj}(\mathcal{I})$ be the maps giving the limit cone on D . Observe that the compositions $f_i \circ \alpha_i: \lim_{\leftarrow} D \rightarrow D'(i)$ form a cone on D' . Thus, there exists a unique factorization through the limit $\lim_{\leftarrow} f: \lim_{\leftarrow} D \rightarrow \lim_{\leftarrow} D'$.

For the colimit claim, let $\beta_i: D'(i) \rightarrow \varinjlim D'(i)$ be the colimit cocone on D' . The composition $\beta_i \circ f_i: D(i) \rightarrow \varinjlim D'$ gives a cocone on D , thus the universal property of the colimit gives a unique map

$$\varinjlim f: \varinjlim D \rightarrow \varinjlim D'$$

which concludes the proof. $\square$

Remark 5.4.4. When $\mathcal{C}$ is complete, we can simply observe that $\varprojlim f$ is nothing more than the functor $\varprojlim$ applied to f , from Proposition 4.3.14. Now, observe that the above statement is slightly more general, since we only need the limits of D and D' to exist, not necessarily all $\mathcal{I}$ -shaped limits. Finally, observe that the proof of Proposition 4.3.14 already contained a proof of the above Lemma.

Proof of Theorem 5.4.2. Let $D: \mathcal{I} \rightarrow \widehat{A}$ be a diagram of presheaves. Define the limit presheaf $X: A^{\text{op}} \rightarrow \underline{\text{Set}}$ as follows. For an object a in A , set $X(a) = \varprojlim (\text{ev}_a \circ D)$, with limit cone given by maps $(\beta^a)_i: X(a) \rightarrow \text{ev}_a \circ D(i) = D(i)(a)$. Now, given a map $\varphi: b \rightarrow a$ in A , compose D with the natural transformation $\text{ev}_\varphi: \text{ev}_a \rightarrow \text{ev}_b$ to obtain the natural transformation $\text{ev}_\varphi D: \text{ev}_a \circ D \rightarrow \text{ev}_b \circ D$. By Lemma 5.4.3 this natural transformation induces a map between the limits. We define $X(\varphi)$ to be that map.

$$X(\varphi) = \varprojlim \text{ev}_\varphi D: \varprojlim \text{ev}_a \circ D \rightarrow \varprojlim \text{ev}_b \circ D.$$

Now, let us construct a cone on D . Let i be an object of $\mathcal{I}$. We want to define a map $\alpha_i: X \rightarrow D(i)$. In order to do so, we must define maps $(\alpha_i)_a: X(a) \rightarrow D(i)(a)$ for all objects a in A . But, by definition $X(a) = \varprojlim \text{ev}_a \circ D$, thus we can just take $(\alpha_i)_a = (\beta^a)_i: X(a) \rightarrow \text{ev}_a \circ D(i) = D(i)(a)$. Now, let us show that $\alpha_i: X \rightarrow D(i)$ is a well-defined map in $\widehat{A}$. To see this, let $\varphi: b \rightarrow a$ be a map in A . We need to show that the following square commutes

$$\begin{array}{ccc} X(a) & \xrightarrow{(\alpha_i)_a} & D(i)(a) \\ X(\varphi) = \varprojlim \text{ev}_\varphi D \downarrow & & \downarrow D(i)(\varphi) = (\text{ev}_\varphi D)_i \\ X(b) & \xrightarrow{(\alpha_i)_b} & D(i)(b) \end{array}$$

But this follows exactly from the definition of $X(\varphi)$ as $\varprojlim \text{ev}_\varphi D$. Hence, $\alpha_i: X \rightarrow D(i)$ is a map in $\widehat{A}$.

Now, we want to show that the α_i form a cone on D . Let $\psi: i \rightarrow j$ be a map in $\mathcal{I}$, we need to show that the following diagram commutes.

$$\begin{array}{ccc} & X & \\ \alpha_i \swarrow & & \searrow \alpha_j \\ D(i) & \xrightarrow{D(\psi)} & D(j) \end{array} \quad (5.4)$$

But, commutativity amounts to the equality between natural transformations

$$D(\psi) \circ \alpha_i = \alpha_j.$$

However, to prove such an equality, it is sufficient to check it at each object a in A . Going back to the definition of $X(a)$ as a limit, we get.

$$D(\psi)_a \circ (\alpha_i)_a = D(\psi)_a \circ (\beta^a)_i = (\beta^a)_j = (\alpha_j)_a.$$

Hence, the diagram (5.4) commutes, and the α_i form a cone.

Finally, let Y be a presheaf on A and $\gamma_i: Y \rightarrow D(i)$ be a collection of maps in $\hat{A}$, for $i \in \text{obj}(\mathcal{I})$ forming a cone. Let us show that this cone factors uniquely through a map $f: Y \rightarrow X$. Let a be an object of A . We construct $f_a: Y(a) \rightarrow X(a)$ as follows. By assumption, the (γ_i) , for $i \in \text{obj}(\mathcal{I})$ form a cone on D , thus if we apply ev_a we see that the $(\gamma_i)_a$ form a cone on $\text{ev}_a \circ D$. Thus, we get a unique factorization $f_a: Y(a) = \text{ev}_a(Y) \rightarrow \lim_{\leftarrow} \text{ev}_a \circ D = X(a)$.

The last thing to check is that f is indeed a map in $\hat{A}$. Let $\varphi: b \rightarrow a$ be a map in A . We need to check commutativity of the following square

$$\begin{array}{ccc} Y(a) & \xrightarrow{f_a} & X(a) = \lim_{\leftarrow} \text{ev}_a \circ D \\ Y(\varphi) \downarrow & & \downarrow X(\varphi) = \lim_{\leftarrow} \text{ev}_\varphi D \\ Y(b) & \xrightarrow{f_b} & X(b) = \lim_{\leftarrow} \text{ev}_b \circ D \end{array}$$

We want to compare two maps, $X(\varphi) \circ f_a$ and $f_b \circ Y(\varphi)$, whose codomain is a limit. Hence, it is sufficient to show that their postcomposition with the $(\beta^b)_i$ are equal, for all $i \in \text{obj}(\mathcal{I})$. Let $i \in \text{obj}(\mathcal{I})$, and consider the diagram

$$\begin{array}{ccccc} & & (\gamma_i)_a & & \\ & \nearrow & & \searrow & \\ Y(a) & \xrightarrow{f_a} & X(a) & \xrightarrow{(\beta^a)_i} & D(i)(a) \\ Y(\varphi) \downarrow & & \downarrow X(\varphi) & & \downarrow D(i)(\varphi) \\ Y(b) & \xrightarrow{f_b} & X(b) & \xrightarrow{(\beta^b)_i} & D(i)(b) \\ & \searrow & & \nearrow & \\ & & (\gamma_i)_b & & \end{array}$$

By assumption, the top and bottom triangles are commutative. Furthermore, since γ_i is a map in $\hat{A}$, the outer rectangle is commutative. Finally, by construction of $X(\varphi)$, the right square is commutative. Thus, we deduce that

$$(\beta^b)_i \circ f_b \circ Y(\varphi) = D(i)(\varphi) \circ (\beta^a)_i \circ f_a = (\beta^b)_i \circ X(\varphi) \circ f_a.$$

Since the above is true for any $i \in \text{obj}(\mathcal{I})$, we must have, $X(\varphi) \circ f_a = f_b \circ Y(\varphi)$. This shows that $f: Y \rightarrow X$ is indeed a factorization in $\hat{A}$. Since it was already unique objectwise, it must be unique, which concludes the proof for the case of the limit.

The proof for the colimit is very similar, and is left as an exercise. $\square$

5.4.2 Kan extensions

Proposition 5.4.5. *Let A be a small category, and $\mathcal{C}$ a cocomplete category. Let $F: A \rightarrow \mathcal{C}$ be any functor. Then there exists a pair of functors, $F^*: \mathcal{C} \rightarrow \widehat{A}$ and $F_!: \widehat{A} \rightarrow \mathcal{C}$ such that*

- *The pair $F_! \dashv F^*$ is an adjoint pair,*
- *there is a natural isomorphism $F_! \circ y \simeq F$.*

Furthermore, the pair $F_! \dashv F^$ is uniquely determined by the above properties (up to natural isomorphism).*

Remark 5.4.6. We can summarize the previous statement through the following diagram.

$$\begin{array}{ccc} A & \xrightarrow{F} & \mathcal{C} \\ y \downarrow & \nearrow F_! & \\ \widehat{A} & & \end{array}$$

Given any functor $F: A \rightarrow \mathcal{C}$, from a small to a cocomplete category, there exists a unique way (up to natural isomorphisms) to extend this functor to a functor $F_!: \widehat{A} \rightarrow \mathcal{C}$. The functor $F_!$ is furthermore the left adjoint to some functor $F^*: \mathcal{C} \rightarrow \widehat{A}$.

Before moving on to the proof, consider the following exercise, that will be used in the proof.

Exercise 75. Let $D: \mathcal{I} \rightarrow \mathcal{C}$ be a functor from a small category $\mathcal{I}$ to some category $\mathcal{C}$, and let X and Y be two objects of $\mathcal{C}$. Write $\text{Hom}_{\mathcal{C}}(D, Y)$ for the functor

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(D, Y): \mathcal{I}^{\text{op}} &\rightarrow \underline{\text{Set}} \\ i &\mapsto \text{Hom}_{\mathcal{C}}(D(i), Y) \\ \varphi: i \rightarrow j &\mapsto \text{Hom}_{\mathcal{C}}(D(\varphi), Y) \end{aligned}$$

and write $\text{Hom}_{\mathcal{C}}(X, D)$ for the functor

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, D): \mathcal{I} &\rightarrow \underline{\text{Set}} \\ i &\mapsto \text{Hom}_{\mathcal{C}}(X, D(i)) \\ \varphi: i \rightarrow j &\mapsto \text{Hom}_{\mathcal{C}}(X, D(\varphi)) \end{aligned}$$

1. Assuming that $\lim_{\leftarrow} D$ exists, show that there exists a bijection, natural in X (hint: this is exactly the unique factorization property of Alternative Definition 4.2.8)

$$\text{Hom}_{\mathcal{C}}(X, \lim_{\leftarrow} D) \simeq \lim_{\leftarrow} \text{Hom}_{\mathcal{C}}(X, D)$$

2. Assuming that $\lim_{\rightarrow} D$ exists, show that there exists a natural bijection:

$$\text{Hom}_{\mathcal{C}}(\lim_{\rightarrow} D, Y) \simeq \lim_{\leftarrow} \text{Hom}_{\mathcal{C}}(D, Y)$$

Proof of Proposition 5.4.5. The easiest functor to defined is the right adjoint $F^*: \mathcal{C} \rightarrow \hat{A}$. Let Y be an object of $\mathcal{C}$. Since we want F^* to be a right adjoint, we must end up with a natural bijection

$$\mathrm{Hom}_{\mathcal{C}}(F_!(X), Y) \simeq \mathrm{Hom}_{\hat{A}}(X, F^*(Y))$$

For X an object in $\hat{A}$. However, by assumption $F_!$ extends F , which means that for any object a in A , we must have $F_!(h_a) \simeq F(a)$. In particular we must have

$$\mathrm{Hom}_{\mathcal{C}}(F(a), Y) \simeq \mathrm{Hom}_{\hat{A}}(h_a, F^*(Y)) \simeq F^*(Y)(a)$$

Where the last bijection is the Yoneda Lemma (Theorem 5.2.3). Thus, we see that the definition for $F^*(Y)$ *has to be*

$$\begin{aligned} F^*(Y): A^{\mathrm{op}} &\rightarrow \underline{\mathrm{Set}} \\ a &\mapsto \mathrm{Hom}_{\mathcal{C}}(F(a), Y) \\ \varphi: a \rightarrow b &\mapsto \mathrm{Hom}_{\mathcal{C}}(F(\varphi), Y): \mathrm{Hom}_{\mathcal{C}}(F(b), Y) \rightarrow \mathrm{Hom}_{\mathcal{C}}(F(a), Y) \end{aligned}$$

Thus, we may write $F^*(Y) = \mathrm{Hom}_{\mathcal{C}}(F(-), Y)$. Given a map $f: X \rightarrow Y$ in $\mathcal{C}$, the map $F^*(f): F^*(X) \rightarrow F^*(Y)$ is then defined as $F^*(f) = \mathrm{Hom}_{\mathcal{C}}(F(-), f)$.

Exercise 76. Make explicit the definition of the functor $F^*: \mathcal{C} \rightarrow \hat{A}$, and check that it is indeed a functor.

Now, to define $F_!: \hat{A} \rightarrow \mathcal{C}$, observe that if it is to be a left adjoint, then it must preserve colimits, by 4.4.17. Thus, if we impose $F_!(h_a) = F(a)$, for all $a \in \mathrm{obj}(A)$, since every objects of $\hat{A}$ is a colimit of representables, this should uniquely define the functor $F_!$. Let us make this intuition explicit. Let X be an object of $\hat{A}$. Recall the category of elements of X , A/X from Definition 5.3.6. Consider the functor $D'_X: A/X \rightarrow A$ sending an object (a, σ) in A/X to the object a of A , and now consider the composition $F \circ D'_X: A/X \rightarrow \mathcal{C}$. Since $\mathcal{C}$ was assumed to be cocomplete, we may define

$$F_!(X) = \lim_{\rightarrow} F \circ D'_X$$

Now, let $f: X \rightarrow Y$ be a map in $\hat{A}$. We need to define $F_!(f): F_!(X) \rightarrow F_!(Y)$. To do so, for $(a, \tau) \in \mathrm{obj}(A/Y)$, let $\beta_{(a, \tau)}: F \circ D'_Y(a, \tau) \rightarrow F_!(Y)$ be the maps forming the colimit cocone. Now, for $(a, \sigma) \in \mathrm{obj}(A/X)$, consider the composition $\gamma_{(a, \sigma)} = \beta_{(a, f_a(\sigma))} \circ \mathrm{Id}_{F(a)}: F \circ D'_X(a, \sigma) = F \circ D'_Y(a, f_a(\sigma)) \rightarrow F_!(Y)$. This forms a cocone on $F \circ D'_X$. Indeed, if $\varphi: (a, \sigma) \rightarrow (b, \sigma')$ is any map in A/X , then $\varphi: (a, f_a(\sigma)) \rightarrow (b, f_b(\sigma'))$ is a map in A/Y . Hence, in that case we have

$$\begin{array}{ccc}
F \circ D'_X(a, \sigma) & \xrightarrow{\text{Id}_{F(a)}} & F \circ D'_Y(a, f_a(\sigma)) \\
\downarrow F(\varphi) & & \downarrow F(\varphi) \\
F \circ D'_X(b, \sigma') & \xrightarrow{\text{Id}_{F(b)}} & F \circ D'_Y(b, f_b(\sigma'))
\end{array}$$

The diagram is completed by two curved arrows from the left column to the right column:

- A top curved arrow from $F \circ D'_X(a, \sigma)$ to $F_!(Y)$ labeled $\gamma_{(a, \sigma)}$.
- A bottom curved arrow from $F \circ D'_X(b, \sigma')$ to $F_!(Y)$ labeled $\gamma_{(b, \sigma')}$.

Exercise 77. Check that the functor $F_1: \hat{A} \rightarrow \mathcal{C}$ is well-defined.

$$\begin{aligned} \mathrm{Hom}_{\widehat{A}}(X, F^*(Y)) &\simeq \mathrm{Hom}_{\widehat{A}}(\lim_{\rightarrow} D_X, F^*(Y)) \\ &\simeq \lim_{\leftarrow} \mathrm{Hom}_{\widehat{A}}(D_X, F^*(Y)) \\ &= \lim_{\leftarrow} \mathrm{Hom}_{\widehat{A}}(y \circ D'_X, F^*(Y)) \\ &\simeq \lim_{\leftarrow} \mathrm{Hom}_{\mathcal{C}}(F \circ D'_X, Y) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(\lim_{\rightarrow} F \circ D'_X, Y) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(F_!(X), Y) \end{aligned}$$

Exercise 78. Check that all of the previous bijections are natural.

☐

Remark 5.4.7. In fact, the concept of Kan extensions is much more general than what we presented here. We now outline the more general notion. Given the data of two functors $F: \mathcal{C} \rightarrow \mathcal{D}$, and $G: \mathcal{C} \rightarrow \mathcal{E}$,

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ G \downarrow & \nearrow L & \nearrow R \\ & \mathcal{E} & \end{array}$$

The left and right Kan extensions of F along G , if they exist, are the data of functors $L, R: \mathcal{E} \rightarrow \mathcal{D}$ and natural transformation $R \circ G \rightarrow F$ and $F \rightarrow L \circ G$, satisfying a universal property among functors $\mathcal{E} \rightarrow \mathcal{D}$. In Proposition 5.4.5, we proved that the left Kan extension existed in a very specific case. Namely, if $G: A \rightarrow \hat{A}$ is the yoneda embedding, and $\mathcal{D}$ is a cocomplete category. Furthermore, we proved that in that case, the natural map $F \rightarrow L \circ G$ was a natural isomorphism and that L admitted a right adjoint.

The full notion of Kan extension is much more general, and ubiquitous. In fact, in his classical book on category theory, MacLane even titled one of his chapter "All concepts are Kan extensions" [Mac71, Chapter X.7]. Note that Kan extensions can be defined (and/or shown to exist) through adjunctions, through certain computation of limits or colimits (as we did in Proposition 5.4.5) or some other way. But conversely, adjunctions and limits can be seen as particular incarnation of the concept of Kan extension. See [Mac71, Chapter X] for more details.

5.4.3 Functors between categories of presheaves

In this last subsection, we use the result of the last section to produce many functors between categories of presheaves.

Definition 5.4.8. Let $F: A \rightarrow B$ be a functor between two small categories. Precomposition with F induces a functor

$$\begin{aligned} F^*: \hat{B} &\rightarrow \hat{A} \\ X &\mapsto X \circ F \\ f: X &\rightarrow Y \mapsto fF: X \circ F \rightarrow Y \circ F \end{aligned}$$

Proposition 5.4.9. Given a functor between small categories $F: A \rightarrow B$, the functor F^* admits both a left adjoint $F_!: \hat{A} \rightarrow \hat{B}$ and a right adjoint $F_*: \hat{A} \rightarrow \hat{B}$.

Proof. We will write $y_A: A \rightarrow \hat{A}$ and $y_B: B \rightarrow \hat{B}$ for the Yoneda embeddings. Consider the composition $F' = y_B \circ F: A \rightarrow \hat{B}$. by Theorem 5.4.2, $\hat{B}$ is a cocomplete category. Hence, we can apply Proposition 5.4.5 to the functor F' to produce a pair of adjoint functors $F'_!: \hat{A} \rightarrow \hat{B}$ and $(F')^*: \hat{B} \rightarrow \hat{A}$. Let us show that $(F')^*$ and F^* are isomorphic functors, which will prove the existence

of a left adjoint to F^* . Let X be an object of $\widehat{B}$, and a an object of A . We compute

$$\begin{aligned} (F')^*(X)(a) &= \text{Hom}_{\widehat{B}}(F'(a), X) \\ &= \text{Hom}_{\widehat{B}}(y_B \circ F(a), X) \\ &= \text{hom}_{\widehat{B}}(h_{F(a)}, X) \\ &\simeq X(F(a)) \\ &= X \circ F(a) \end{aligned}$$

where the only non-trivial isomorphism is given by the yoneda lemma. The same computation for a map $\varphi: a \rightarrow b$ in A shows that the above bijection assemble to give an isomorphism $\alpha_X: (F')^*(X) \simeq F^*(X)$. Now, to see that α is a natural transformation, recall that the isomorphism appearing in the yoneda lemma (Theorem 5.2.3) is natural in both variables. Thus $(F')^* \simeq F^*$, and F^* admits a left adjoint.

Now, consider the composition $G = F^* \circ y_B: B \rightarrow \widehat{A}$. As we just did before, we can apply Proposition 5.4.5 to G , to produce a pair of adjoint functors, $G_! \dashv G^*$, with $G_!: \widehat{B} \rightarrow \widehat{A}$ and $G^*: \widehat{A} \rightarrow \widehat{B}$. It will be enough to prove that $G_! \simeq F^*$ to prove that F^* admits a right adjoint. We define a natural isomorphism $\beta: G_! \rightarrow F^*$. Let X be an object of $\widehat{B}$, we need to define for each object a of A a bijection

$$(\beta_X)_a: G_!(X)(a) \rightarrow F^*(X)(a)$$

But now, by definition of $G_!$ together with Theorem 5.4.2, we have a natural isomorphism

$$G_!(X)(a) \simeq \lim_{\rightarrow} \text{ev}_a \circ G \circ D_X$$

Where $D_X: B/X \rightarrow B$ is the diagram of elements of X . Now, observe that by definition $G = F^* \circ y_B$, and $\text{ev}_a \circ F^* = \text{ev}_{F(a)}$. Thus, we have a natural isomorphism

$$G_!(X)(a) \simeq \lim_{\rightarrow} \text{ev}_{F(a)} \circ y_B \circ D_X$$

And the codomain of the above natural isomorphism is naturally isomorphic to $X(F(a)) = F^*(X)(a)$ by Proposition 5.3.8 together with Theorem 5.4.2. Since all of the above isomorphisms are natural with respect to both X and a , we conclude that their composition $(\beta_X)_a: G_!(X)(a) \rightarrow F^*(X)(a)$ assemble into a natural isomorphism $\beta: G_! \rightarrow F^*$. This concludes the proof. $\square$

Exercise 79. Let A and B be two small categories and $F: A \rightarrow B$ and $G: B \rightarrow A$ be a pair of adjoint functors $F \dashv G$.

1. Show that $F^* \dashv G^*$ form an adjoint pair. (Hint: reason carefully with units and counits of the adjunction $F \dashv G$).
2. Deduce that there is a chain of adjunctions and natural isomorphisms.

$$F_! \dashv F^* \simeq G_! \dashv F_* \simeq G^* \dashv G_*$$

In particular, deduce that F_* has a right adjoint and that $G_!$ has a left adjoint.

Remark 5.4.10. In a lot of contexts, we may only care about the adjoint pair $F_! \dashv F^*$. However, even in those case, knowing that a right adjoint to F^* exist may prove useful. First of all, because it implies that F^* must preserve *both* limits and colimits (Propositions 4.3.15 and 4.4.17), and because it makes statements such as the one in the previous exercise less surprising and easier to parse. Note however that in those cases where we do not necessarily care about the right adjoint to F^* it may happen that its left adjoint is denoted F_* , instead of $F_!$ (and its right adjoint is not even named) which may lead to great confusion if one is not vigilant.

Chapter 6

Simplicial sets

Bibliography

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